Cerf Diagrams and Hatcher-Wagoner Invariants for Barbell Maps

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Dedicated to my beloved, Xiayu Tan,
for the beauty of her research
and the warmth of her heart.

— Yifei Fan

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Editor's Preface

It is with a heart full of excitement and pride that I present this work. I have had the privilege of witnessing Xiayu's remarkable transformation — from a budding topology enthusiast to a seasoned researcher capable of navigating the vast sea of literature to find truth.

Throughout this journey, I saw her moments of pure joy upon mastering a complex paper, as well as her frustrations when a seemingly brilliant construction turned out to be trivial. The day she told me, "I've done it," I felt a profound sense of happiness for her. She has truly come into her own as a researcher.

As a gift to celebrate her first-ever paper, I have prepared this work in the form of a journal article. It is my sincere wish that Xiayu will go on to publish in top-tier journals and continue to tackle even more challenging problems in the world of mathematics.

The Editor-in-Chief

Introduction

Given a smooth oriented manifold X, let $\mathcal{P}(X)$ be the pseudo-isotopy group of X defined by

$$\mathcal{P}(X) := \{f \in \text{Diff}(X \times I) | f_{\partial X \times I \cup X \times 0} = \text{id}\}$$

The pseudo-isotopy group is closely related to the diffeomorphism group of X by $\mathcal{P}(X) \xrightarrow{F} \text{Diff}(X, \partial) : f \to F_f = f|_{X \times 1}$, in that case we call that f results in F_f.

Hatcher and Wagoner first studied $\pi_0\mathcal{P}(X)$ in their collected work [1] and it was later studied by Igusa in [2]. They found two invariants for the group, namely,

$$\Sigma : \pi_0(P) \to \text{Wh}_2(\pi_1 X) \text{ and } \Theta : \ker \Sigma \to \text{Wh}_1(\pi_1 X; \mathbb{Z}_2 \times \pi_2 X)/\chi(K_3\mathbb{Z}[\pi_1 X])$$

where we have $\text{Wh}_1(\pi_1 X; \mathbb{Z}_2 \times \pi_2 X) = (\mathbb{Z}_2 \times \pi_2 M)[\pi_1 M]/(\beta \cdot [1], \alpha \cdot [\sigma] - \alpha^\tau \cdot [\tau\sigma\tau^{-1}], \alpha, \beta \in \mathbb{Z}_2 \times \pi_2 M, \tau, \sigma \in \pi_1 M)$. These two maps are always well-defined whenever $\text{dim}X \geq 4$, and they showed that Σ is always surjective when $\text{dim}X \geq 5$, and Θ is surjective when $\text{dim}X \geq 5$ and bijective when $\text{dim}X \geq 6$. In [3] Singh studied the partial images of Σ and Θ in dimension 4 and showed that both Σ and Θ are stably surjective.

The pseudo-isotopy group is closely related to the diffeomorphism group, for there is a natural fiber bundle

$$\mathcal{J} = \text{Diff}(X \times I, \partial) \to \mathcal{P}(X) \xrightarrow{F} \text{Diff}_{PI}(X, \partial)$$

where $\text{Diff}_{PI}(X, \partial) \subset \text{Diff}(X, \partial)$ denotes all the diffeomorphisms of X which are pseudo-isotopic to identity. Thus we have *induced* Hatcher-Wagoner invariants

$$\Sigma : \pi_0 \text{Diff}_{PI}(X, \partial) \to \text{Wh}_2(\pi_1 X)/\Sigma(\mathcal{J})$$

$$\Theta : \pi_0 F(\ker \Sigma) \to \text{Wh}_1(\pi_1 X; \mathbb{Z}_2 \times \pi_2 X)/(\Theta(\mathcal{J} \cap \ker \Sigma) + \chi(K_3\mathbb{Z}[\pi_1 X]))$$

on the diffeomorphism group of X, where $\pi_0 F(\ker \Sigma) \subset \pi_0 \text{Diff}_{PI}(X, \partial)$

Recently, Gabai, Budney, Gay and Hartman constructed the 4-dimensional barbell diffeomorphism (a specific nontrivial element in $\pi_0 \text{Diff}(\mathcal{B}^4_{2,2}, \partial)$ following the notations in [4]) and studied the implanted barbell diffeomorphisms in a general X^4, especially in $S^1 \times D^3$ ([5]) and S^4 ([6]). One of the remarkable results is that Budney and Gabai constructed an infinitely generated subset in $\pi_0(\text{Diff}(S^1 \times D^3, \partial)/\text{Diff}(D^4, \partial))$, more explicitly, an inclusion

$$\bigoplus_{k \geq 4 \in \mathbb{N}} \mathbb{Z} \to \pi_0(\text{Diff}(S^1 \times D^3, \partial)/\text{Diff}(D^4, \partial)) : 1_k \to \delta_k$$

where $\{\delta_k\}_{k \in \mathbb{N}}$ are specific implanted barbell diffeomorphisms in $S^1 \times D^3$ which we will carefully describe later in this paper.

Suppose we have an implanted barbell in a general X^4, i.e. $\beta : S^2 \times D^2 \natural S^2 \times D^2 \hookrightarrow X$, and therefore an implanted barbell diffeomorphism in $\mathrm{Diff}(X, \partial)$, it is known that (see `[4, Proposition 2.6]`) if one of the implanted core spheres is unknotted in X^4, then the implanted barbell diffeomorphism is pseudo-isotopic to identity. One natural question is:

Question. How to compute the induced Hatcher-Wagoner invariants for the given half-unknotted implanted barbell diffeomorphism?

Once we get an $f_\beta \in \pi_0\mathcal{P}$ resulting in that implanted barbell diffeomorphism, $\Sigma(f_\beta), \Theta(f_\beta)$ are representatives of the two induced Hatcher-Wagoner invariants. In this article, we'll construct two pseudo-isotopies $g_\beta, f_\beta \in \pi_0\mathcal{P}$, both resulting in the implanted barbell diffeomorphism with respect to β. g_β has a Cerf diagram containing a single eye of (1,2)-handle pair and f_β has a Cerf diagram containing a single eye of (2,3)-handle pair. The main result will give an explicit formula for the Hatcher-Wagoner invariants of f_β, which is stated as follows:

Theorem 0.1. *For a half-unknotted implanted barbell $\beta = (R_0, S, \gamma)$ with $S = \partial\beta_0^\bullet$ where $\beta_0^\bullet : D^3 \hookrightarrow M$, by finger-pushing R_0 along the arc, we can make γ short enough such that $int(\beta_0^\bullet) \cap \gamma = \emptyset$. Now suppose that $\beta_0^\bullet \cap R_0 = \bigsqcup_{i=1}^k S_i^1$. Choose $p_i \in S_i^1$ and let $*_0 = \gamma \cap S$ be the base point. For each i, find a path $\delta_i^B \subset \beta = (R_0, S, \gamma)$ which is a path from $*_0$ to $p_i \in S_i^1$. Also, for each i, S_i^1 divides R_0 into two embedded disks D_i and D_i', where D_i' is the one connected to the arc γ. Let $D_i^B = D_i$ (see* Figure 1 *for an illustration). Then there is a pseudo-isotopy $f_\beta \in \pi_0\mathcal{P}$ resulting in that implanted barbell diffeomorphism with the Cerf diagram of f_β being a single eye of (2,3)-handle pair such that:*

$$\Sigma(f_\beta) = 0, \Theta(f_\beta) = \sum_{i=1,\dots,k} (0, [D_i^B]^{\delta_i^B}) \cdot [\delta_i^B]$$

Here we identify $\pi_i(M, *_0)$ with $\pi_i(M, \beta_0^\bullet)$ so that $[D_i] \in \pi_2(M, \beta_0^\bullet)$ and $\delta_i^B \in \pi_1(M, \beta_0^\bullet)$.

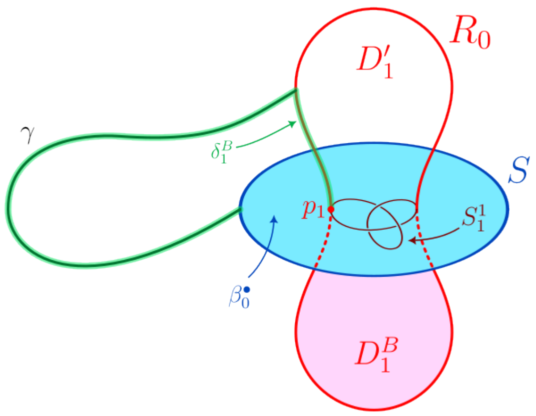

Figure 1 / A general barbell and data needed to compute Hatcher-Wagoner invariants

Remark 0.2. In particular, when the given barbell $\beta = (R_0, S, \gamma)$ with $S = \partial\beta_0^\bullet$ and $\gamma : [0,1] \to M, \gamma(0) = *_0 = \gamma \cap S$ satisfies $R_0 \cap \beta_0^\bullet = \emptyset$, suppose $\gamma|_{(0,1)} \cap \beta_0^\bullet = \{p_i\}_{i=1}^k$ with $p_i = \gamma(t_i)$, then the $f_\beta \in \pi_0\mathcal{P}$ we constructed in the theorem satisfies:

$$\Sigma(f_\beta) = 0, \Theta(f_\beta) = (0, [R_0^\gamma]) \sum_{i=1}^{k} [\gamma_i]$$

where $\gamma_i := \gamma|_{[0,t_i]} \in \pi_1(M, \beta_0^\bullet)$ and $[R_0^\gamma] \in \pi_2(M, \beta_0^\bullet)$ is obtained by pulling R_0 back to $*_0$ along γ. ⌟

Remark 0.3. Along the way we also compute the Hatcher-Wagoner invariants for g_β: $\Sigma(g_\beta) = 0, \Theta(g_\beta) = -\bar{\Theta}(f_\beta)$, where $\bar{\cdot}$ is an involution on $\mathrm{Wh}_1(\pi_1, \mathbb{Z}_2 \times \pi_2)$ which is described in both `[1]` and `[3, section 9.1]`. We review the involution here:

$$\bar{\cdot} : (n, \sigma) \cdot [\gamma] \to (n + w_2^X(\sigma), -w_1^X(\gamma)\sigma^{\gamma^{-1}}) \cdot [\gamma^{-1}],$$

where $w_2^X(\sigma) \in \{0, 1\}$ is the second Stiefel-Whitney class of the normal bundle of σ in X, and $w_1^X(\gamma) \in \{-1, 1\}$ is the first Stiefel-Whitney class of the normal bundle of γ in X. ⌟

Remark 0.4. In particular, all $\{f_{\delta_k}\}_{k\in\mathbb{N}}$ in $\mathcal{P}(S^1 \times D^3)$ lie in the kernel of Σ and Θ, this answers a question raised by Powell in `[7, Question 12.6]`. But $\{\delta_k, k \geq 4\}$ are nontrivial diffeomorphisms of $S^1 \times D^3$, so $[f_{\delta_k}, k \geq 4] \neq 0 \in \pi_0\mathcal{P}(S^1 \times D^3)$, which equivalently means that $\{f_{\delta_k}, k \geq 4\}$ are nontrivial pseudo-isotopies of $S^1 \times D^3$ which can not be detected by Hatcher-Wagoner invariants. ⌟

We generalize the calculation to *half-unknotted immersed barbell diffeomorphisms* which will be defined in Chapter 5. The result is completely the same:

Theorem 0.5. *For a half-unknotted immersed barbell $\beta = (R_0, S, \gamma)$ with $\beta_0^\bullet : D^3 \hookrightarrow M, S = \partial\beta_0^\bullet$, perturb self-intersections of R_0 away from $\beta_0^\bullet$. By finger-pushing R_0 along arc, we can make γ short enough such that int$(\beta_0^\bullet) \cap \gamma = \emptyset$. Now suppose that $\beta_0^\bullet \cap R_0 = \bigsqcup_{i=1}^k S_i^1$. Choose $p_i \in S_i^1$ and let $*_0 = \gamma \cap S$ be the base point. For each i, find a path $\delta_i^B \subset \beta = (R_0, S, \gamma)$ which is a path from $*_0$ to $p_i \in S_i^1$. Also, for each i, S_i^1 divides R_0 into two embedded disks D_i and D_i', where D_i' is the one connected to the arc γ. Let $D_i^B = D_i$. Then the $f_\beta \in \pi_0\mathcal{P}$ we constructed which results in the immersed barbell diffeomorphism w.r.t. β satisfies:*

$$\Theta(f_\beta) = \sum_{i=1,\dots,k} (0, [D_i^B]^{\delta_i^B}) \cdot [\delta_i^B]$$

Here we identify $\pi_i(M, *_0)$ with $\pi_i(M, \beta_0^\bullet)$ so that $[D_i] \in \pi_2(M, \beta_0^\bullet)$ and $\delta_i^B \in \pi_1(M, \beta_0^\bullet)$.

From the main theorem we deduce some corollaries:

Corollary 0.6. *For any $\sigma \in \pi_2 M$, $\forall \alpha \in \pi_1 M$, there is a half-unknotted immersed barbell $\beta = (R, S, \gamma)$ and $f_\beta \in \ker\Sigma \subset \pi_0\mathcal{P}$ with its Cerf diagram being a single eye of (2,3)-handle pair resulting in the immersed barbell diffeomorphism w.r.t. β such that $\Theta(f_\beta) = (0, \sigma) \cdot [\alpha]$.*

In particular, using results deduced by Singh in [3] where Singh showed that for $M = (X_1 \# X_2) \times I$ with X_i a closed, oriented, aspherical 3-manifold, there's $K \subset \pi_0 \operatorname{Diff}_{PI}(M, \partial)$ and a surjection $K \to \bigoplus_{i \in \mathbb{N}} \mathbb{Z}$, we further show that we can let

$$K = \langle \text{implanted half-unknotted barbell diffeomorphisms} \rangle,$$

that is:

Corollary 0.7. *Let $M = (X_1 \# X_2) \times I$ with X_i closed, orientable, aspherical 3-manifold. In this case all $\sigma \in \pi_2 M = \mathbb{Z}[\pi_1 X_1 * \pi_1 X_2]$ can be realized by embedded S^2 with $w_2^M(\sigma) = 0$. Then*

$$\langle \textit{implanted half-unknotted barbell diffeomorphisms} \rangle \subset \pi_0 \operatorname{Diff}_{PI}(M, \partial)$$

is infinitely generated and of infinite $\mathbb{Z}$-rank.

The paper is organized as follows: In Chapter 1 we recall the Cerf theory and parametrised handle constructions version of it, especially we explain how a loop of handle constructions of $X \times I$ can result in a pseudo-isotopy $f \in \pi_0 \mathcal{P}$ of X, and how to see the resulting $F_f \in \operatorname{Diff}(X, \partial)$ directly from the loop of handle constructions. In Chapter 2 we recall the general notion of Montesino twins in X^4 and the resulting twin twists in $\operatorname{Diff}_0(X, \partial)$ which were studied for $X = S^4$ in [6]. Then we show that if one of the spheres in the Montesino twins is unknotted, then $\exists f \in \mathcal{P}(X)$ resulting in that twin twist with Cerf diagram being a loop of a single cancelling (1,2)-handle pair. Then we show that any implanted barbell diffeomorphism is a twin twist, so together with the above result, for a specific implanted barbell β, we find $g_\beta \in \mathcal{P}(X)$ with Cerf diagram being a loop of a single cancelling (1,2)-handle pair resulting in that implanted barbell diffeomorphism. In Chapter 3 we follow the essential lemma [6, Lemma 17] to develop a technique of *one parameter version of 0-framed and dotted replacement* to change the pseudo-isotopy $g_\beta \in \pi_0 \mathcal{P}$ with Cerf diagram being the above loop of cancelling (1,2)-handle pair to $f_\beta \in \pi_0 \mathcal{P}$ with Cerf diagram being a loop of cancelling (2,3)-handle pair. Therefore we get a loop of cancelling (2,3)-handle pair resulting in implanted barbell diffeomorphism of β. In Chapter 4 we use the technology in Chapter 3 to complete the computations of second Hatcher-Wagoner invariant for a half-unknotted implanted barbell diffeomorphism. In Chapter 5 we generalize to *immersed barbell diffeomorphisms* and compute the induced Hatcher-Wagoner invariants for a half-unknotted immersed barbell diffeomorphism. As corollaries, we deduce some realization results on what induced Hatcher-Wagoner invariants can be realized by immersed half-unknotted barbell diffeomorphisms.

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The author would like to thank Jean Cerf, Allen Hatcher and John Wagoner for developing the excellent theory about pseudo-isotopy. She also thanks David Gabai, David Gay, Daniel

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The work was carried out during the author's fourth year in Qiuzhen College, Tsinghua University, and thus she expresses her deep gratitude to the institution.

1 Cerf theory and parameterised handle constructions version

We have defined the pseudo-isotopy group $\mathcal{P}(X)$ and want to study $\pi_0\mathcal{P}(X)$. Cerf discovered a great correspondence:

$$\pi_0\mathcal{P} \cong \pi_1(\mathcal{F}, \mathcal{E})$$

where $\mathcal{F} = \mathcal{F}(X) := \{f : X \times I \to I | f = \text{standard projection on a neighborhood of } \partial(X \times I)\}$ and $\mathcal{E} = \mathcal{E}(X) \subset \mathcal{F}(X)$ contains all such f that have no critical points at all. The explicit correspondence is: Given $f \in \mathcal{P}(X)$, let p be the standard projection from $X \times I$ to I. Then $p \circ f \in \mathcal{E}(X)$ since f is a diffeomorphism of $X \times I$. It is not hard to show that $\mathcal{F}(X)$ is contractible, therefore we can choose a path $\gamma = \{\gamma_t : t \in I\}$ in $\mathcal{F}$ with $\gamma_0 = p, \gamma_1 = p \circ f$ both in $\mathcal{E}$. Then γ is the corresponding element in $\pi_1(\mathcal{F}, \mathcal{E})$. Conversely, given $f_t : X \times I \to I$ starting at $f_0 = p$ and ending at $f_1 = q \in \mathcal{E}$, consider the gradient vector field V_q of q on $X \times I$, integrating V_q gives a diffeomorphism on $X \times I : (x, t) \to \phi_{V_q}(x, 0)(t)$.

Then we reduce the question to the study of *one parameter family of functions* on $X \times I$, namely, $f_t : X \times I \to I$ with f_0, f_1 no critical points at all. Like what we did in Morse theory, we can perturb the path fixing f_0, f_1 such that for all but finitely many $t \in I$, f_t is a Morse function with ordered, distinct critical values. At the exceptional points, there appears either an $(i, i+1)$-birth/death point (two critical points of index i and $i+1$ have the same critical value, but by the ordering condition, they must cancel in one direction) or an (i, i)-crossing point (two non-degenerate critical points of same index i with crossing critical values). The Cerf diagram records the changes of critical values in $f_t : X \times I$. An example is shown in Figure 1.1.

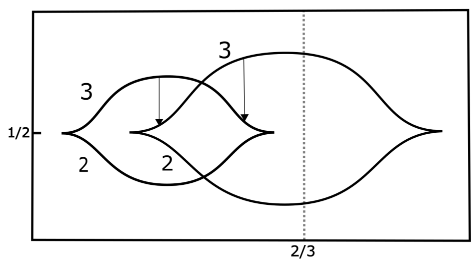

Figure 1.1 / This is a Cerf diagram with 2 eyes, each eye begin from the birth to the end. All births and ends happen in $g_t^{-1}(1/2)$. Except the four birth/death points, every g_t is Morse. For example, $g_{2/3}$ has two non-degenerate critical points of index 2 and 3. There are two crossing points in the diagram. Every arrow denotes a handle slide between critical points with the same index.

In this paper we are trying to find a Cerf diagram for f_β with only one eye of index 2 and 3. Then let $g_t, t \in I$ be the corresponding one parameter family of functions on $X \times I$ with $g_0 = p, g_1 = p \circ f_\beta$. If we successfully find such a family g_t with such a Cerf diagram, then the only things happen here are as follows: For some small ϵ, g_ϵ births a degenerate critical point(standard local model is $g_\epsilon = x^3 - y_1^2 - y_2^2 + y_3^2 + y_4^2$ near the origin $p = (0, 0, 0, 0, 0)$ with $g_\epsilon(p) = 1/2$), and for $t > \epsilon$, the critical point splits into two non-degenerate critical points of indices 2 and 3(standard local model is $g_{\epsilon+s} = x^3 - sx - y_1^2 - y_2^2 + y_3^2 + y_4^2$). Right after the birth point at $g_\epsilon^{-1}(1/2)$, the level set is $g_{\epsilon+s}^{-1}(1/2) = X\#S^2 \times S^2$, with the dual sphere of the unique 2-handle being $* \times S^2$ and the attaching sphere of the unique 3-handle being $S^2 \times *$. Then the two spheres isotope in $X\#S^2 \times S^2$ during the interval $t \in (\epsilon, 1-\epsilon)$. They may intersect each other at more than one point during the isotopy, but finally, right before $t = 1-\epsilon$, they return to the dual position, i.e. intersect at exactly one point. Then general Morse theory tells that the (2,3)-handle pair can be cancelled, so a death occurs at $t = 1-\epsilon$. But via an isotopy, we can assume one of the spheres is fixed throughout. In short, the $\{g_t, t \in I\} \in \pi_1(\mathcal{F}, \mathcal{E})$ is completely governed by an element in $\pi_1(\mathrm{Emb}(S^2 \times D^2, X\#S^2 \times S^2), \mathrm{Emb}_0(S^2 \times D^2, X\#S^2 \times S^2))$ where $\mathrm{Emb}_0(S^2 \times D^2, X\#S^2 \times S^2)$ denotes all the embedded $S^2 \times D^2$ such that $S^2 \times 0$ transversely intersect $* \times S^2$ at a single point.

A similar story holds for a Cerf diagram with only one eye of index 1 and 2. To be explicit, $\{g_t, t \in I\} \in \pi_1(\mathcal{F}, \mathcal{E})$ is completely governed by an element in $\pi_1(\mathrm{Emb}(S^1 \times D^3, X\#S^1 \times S^3), \mathrm{Emb}_0(S^1 \times D^3, X\#S^1 \times S^3))$ with similar notation for $\mathrm{Emb}_0(S^1 \times D^3, X\#S^1 \times S^3)$.

In particular, whenever we have a loop of framed embedded S^i in $X\#S^i \times S^{n-i}$, which means the element lies not only in the relative π_1, but also in $\pi_1(\mathrm{Emb}(S^i \times D^{n-i}, X\#S^i \times S^{n-i}), *)$, tracing the flow lines it is not hard to show that the resulting diffeomorphism on X is just the following composition:

$$\pi_1(\mathrm{Emb}(S^i \times D^{n-i}, X\#S^i \times S^{n-i}), *) \to \pi_0 \mathrm{Diff}(X \setminus \nu S^{n-i-1}, \partial) \xrightarrow{\bigcup \mathrm{id}_{\nu S^{n-i-1}}} \pi_0 \mathrm{Diff}(X, \partial)$$

where the first map is the isotopy extension at $t = 1$ and removing $\nu S^i = \nu(S^i \times *)$ in $X\#S^i \times S^{n-i}$, which is diffeomorphic to $X \setminus \nu S^{n-i-1}$, where the S^{n-i-1} denotes the dual sphere of the $(i+1)$-handle.

Just like *"Building a manifold X from a Morse function is equivalent to finding a handle decomposition for X"*, we have an equivalent version for pseudo-isotopy: *Finding a one-parameter family of functions on $X \times I$ is equivalent to finding a one-parameter family of handle decompositions for $X \times I$.*

Thus if we have a loop $\mathcal{H} = \{H_t, t \in I | H_0 = H_1\}$ of handle decompositions of $X \times I$ which starts and ends at a standard cancelling position which can be made null, by the above statement, this corresponds to an element in $\pi_1(\mathcal{F}, \mathcal{E}) = \pi_0\mathcal{P}$, the resulting manifold is Z_t which is diffeomorphic to $X \times I$. Since $H_0 = H_1$, we have a natural way to identify Z_0 with Z_1 (since they are totally the same manifold). Then we build a cobordism from $S^1 \times X$ to $Y = X \times I/((x,0) \sim (f_{\mathcal{H}}(x), 1))$ where $f_{\mathcal{H}}$ is the resulting diffeomorphism on X with respect to $\mathcal{H} \in \pi_1(\mathcal{F}, \mathcal{E})$. In handlebody language, we describe what $f_{\mathcal{H}}$ is: Each H_t gives a series of surgeries $X = X_{0,t} \to X_{1,t} \to ... \to X_{n,t}$, $X_{n,t}$ is diffeomorphic to X but not in a natural way. But again, since $H_0 = H_1$, we have $X_{n,0} = X_{n,1} \cong X$. But as t varies, since H_t moves smoothly, every attaching region moves smoothly, thus we have a diffeomorphism $\phi_{m,t} : X_{m,0} \to X_{m,t}, \forall m, t$. In particular $\phi_{n,1} = f_{\mathcal{H}} : X = X_{n,0} \to X_{n,1} = X$ is the desired diffeomorphism on X.

In particular, when $\mathcal{H}$ is a loop of a single $(i, i+1)$-cancelling pair, that is, $Z_t = X \times I \cup h_{i,t} \cup h_{i+1,t}$. Since $h_{i,0}$ and $h_{i+1,0}$ form a standard cancelling pair, the attaching sphere of $h_{i,0}$ must be unknotted. Then let $X_{1,0} = (X \setminus \nu S^{i-1}) \cup D^i \times S^{n-i} = X \# S^i \times S^{n-i}$ be the manifold after doing surgery on h_i. Up to an isotopy, we can assume $h_{i,t} = h_{i,0}$ throughout, so $X_{1,t} = X_{1,0} = X \# S^i \times S^{n-i}$. The only t-dependent data is the attaching region of $h_{i+1,t}$, which corresponds to $\pi_1(\text{Emb}(S^i \times D^{n-i}, X_{1,0}), *)$, then one can directly see that the resulting $f_{\mathcal{H}} : X = X_{2,0} \to X_{2,1} = X$ is the composition map mentioned above.

2 Twin twists and relation with barbell diffeomorphisms

First we derive a standard model for Montesino twins: Consider $X = S^1 \times S^1 \times D^2$ with boundary $\partial X = S^1_l \times S^1_S \times S^1_R$, where $S^1_l = S^1 \times 1 \times 1$, $S^1_S = -1 \times S^1 \times 1$, $S^1_R = -1 \times 1 \times \partial D^2$. Then we do surgery on $\nu(1 \times S^1 \times 0)$, and let $X' = (X \setminus \nu(1 \times S^1 \times 0)) \cup D^2 \times S^2$ be the resulting manifold. Then the core $T_R = (S^1 \times S^1 \times 0)$ of X becomes $R = (T_R \setminus I \times S^1) \cup \partial I \times D^2$, which is a 2-sphere with normal sphere S^1_R, and $S = 0 \times S^2 \subset D^2 \times S^2$ is another 2-sphere with normal sphere S^1_S. Moreover, R intersects with S transversely at 2 points. The resulting manifold X' is just a tubular neighborhood of $R \cup S$ with the same boundary $\partial X' = \partial X = S^1_l \times S^1_S \times S^1_R$. Centered at $* = (-1, 1, 1)$, S^1_l is homotopic to a longitude of X', that is, $\pi_1(\partial X', *) \to \pi_1(X', *) \cong \pi_1(S^2 \vee S^2 \vee S^1, *) = \mathbb{Z}$ sends $[S^1_l]$ to 1 after a suitable orientation. An easy way to see that is, $\pi_1(\partial X, *) \to \pi_1(X, *)$ simply kills S^1_R and sends others via identity, and surgering X to X' simply kills S^1_S since it's a parallel copy of $1 \times S^1 \times 0$, that means $\pi_1(X') = \pi_1(X)/[S^1_S] = \mathbb{Z}[S^1_l]$. Moreover, a simple way to construct a longitude in X' is to consider $\{p, q\} = R \cap S$, and choose $\gamma_R \subset R, \gamma_S \subset S$ connecting p and q, then the resulting $\gamma_R \cup_{p,q} \gamma_S$ is a longitude of X'.

A standard twin twist is a Dehn twist near the boundary of X', that is, choose a neighborhood of $\partial X' = I \times S^1_l \times S^1_S \times S^1_R$, and let $\phi_D \in \mathrm{Diff}(I \times S^1_l, \partial)$ be the standard Dehn twist, then twin twist τ is identity-extension of $\phi_D \times \mathrm{id} \times \mathrm{id} \in \mathrm{Diff}(\nu(\partial X'), \partial)$ in $\mathrm{Diff}(X', \partial)$.

We provide a more natural way to construct the twin twist τ: Consider a loop of S^1 in X: $\gamma_t : S^1 \to e^{2\pi i t} \times S^1 \times 0$, the isotopy extension of γ_t at $t = 1$ induces a diffeomorphism on X which fixes γ_0. Moreover, it can be made to fix the whole core T_R, that is, it induces a diffeomorphism on $X \setminus \nu T_R = \nu(\partial X)$, which is exactly $\phi_D \times \mathrm{id} \times \mathrm{id}$. Therefore τ is the induced diffeomorphism after surgering γ_0.

Definition 2.1. A *Montesino twin* in M^4 is a pair $W = (R, S)$ where R, S are two embedded S^2 in M with trivial normal bundle and R intersects S transversely at 2 points. Equivalently, a Montesino twin in M^4 is an embedding $i_W : X' \hookrightarrow M$. The twin twist induced by W is just the implanted diffeomorphism τ, denoted by τ_W.

Any twin twist is induced by a *parameterised surgery of index one* (see [8] for details) we will briefly describe below:

Definition 2.2. Given a framed embedded S^2 in M^4, that is, $\nu S : S^2 \times D^2 \to M^4$, let $M_{\nu S} := (M \setminus \nu S) \cup D^3 \times S^1$ be the manifold obtained by performing surgery on S. We say a diffeomorphism ϕ of M is induced by a *parameterised surgery of index one* if ϕ is in the image of $\mathrm{ps}_{\nu S}$:

$$\mathrm{ps}_{\nu S} : \pi_1(\mathrm{Emb}(\nu S^1, M_{\nu S}), i_0) \to \pi_0 \mathrm{Diff}(M \setminus \nu S, \partial) \xrightarrow{\cup \mathrm{id}_{\nu S}} \pi_0 \mathrm{Diff}(M, \partial)$$

where $i_0 = 0 \times S^1 \subset M_{\nu S}$ with natural framing is the base point of the embedding space and the first map is induced by isotopy extension.

Proposition 2.3. *For any Montesino twin $W = (R, S)$ in M, the twin twist τ_W is induced by a* parameterised surgery of index one *on S.*

Proof. By the standard model of Montesino twins we have constructed, we do surgery on S we get an embedding $i : X = S^1 \times S^1 \times D^2 = \nu T^2 \to M_{\nu S}$ with $i_0 = i(1 \times S^1 \times 0)$. By the second way we describe twin twist τ, $\tau_W = \mathrm{ps}_{\nu S}(\gamma_t)$ where $\gamma_t : S^1 \to i(e^{2\pi i t} \times S^1 \times 0) \subset M_{\nu S}$. □

Proposition 2.4. *When $W = (R, S)$ with S unknotted, then there exists a loop $\mathcal{H} \in \pi_0 \mathcal{P}$ of handle decompositions of $M \times I$ with a single (1,2)-cancelling pair resulting in τ_W. In particular, τ_W is pseudo-isotopic to identity with the corresponding Cerf diagram being a single eye of (1,2)-handle pair.*

Proof. When S is an unknotted S^2, $M_{\nu S} = M \# S^3 \times S^1 = M \# S^1 \times S^3$, this can be regarded as attaching a trivial one handle. Then Prop 3.3 proved that τ_W is induced by an element $\gamma \in \pi_1(\mathrm{Emb}(\nu S^1, M \# S^1 \times S^3), i_0)$ where $i_0 = S^1 \times * \in S^1 \times S^3$. Then $\tau_W = \phi_\gamma(1)|_{M \# S^1 \times S^3 \setminus \nu S^1} \bigcup \mathrm{id}_{\nu S^2} = f_{\mathcal{H}}$ where $\phi_\gamma(t)$ denotes the time t isotopy extension on $M \# S^1 \times S^3$. □

Then we recall the definition of an (implanted) barbell diffeomorphism and prove that any implanted barbell diffeomorphism is a special twin twist.

Definition 2.5. The *standard barbell diffeomorphism* is defined as follows: Consider D^4 and 2 disjoint properly embedded arcs I_1, I_2 in it. Let I_1 rotate around a normal sphere $S^2 \times * \subset SN(I_2) = S^2 \times I_2$ and comes back. This isotopy extension induces an element in $\mathrm{Diff}(D^4 \setminus (\nu I_1 \cup \nu I_2) = S^2 \times D^2 \natural S^2 \times D^2, \partial)$. The *implanted barbell diffeomorphism* is obtained by implanting a barbell $\beta = (S_1, S_2, \gamma)$, i.e. S_1, S_2 two framed embedded S^2 with an arc connecting them, and extending the barbell diffeomorphism by id on $M \setminus \nu\beta$.

Proposition 2.6. *Let $\beta = (R_0, S, \gamma)$ be an implanted barbell in M. Finger-push R_0 along γ into S to get another sphere R, then R intersects S transversely at 2 points. Then the implanted barbell diffeomorphism by β is isotopic to twin twist τ_W where $W = (R, S_0)$.*

Proof. We first describe τ_W with $W = (R, S)$: After surgering along S_0, we get an embedding $\nu T_R \to M_{\nu S}$ where T_R is a torus which has a framing induced by νR_0, this corresponds to an element $[\nu T_R]$ in $\pi_1(\mathrm{Emb}(\nu S^1, M_{\nu S}), *)$, then we have:

$$\begin{array}{ccc} \pi_1(\mathrm{Emb}(\nu S^1, M_{\nu S}), *) & \longrightarrow & \pi_0 \mathrm{Diff}(M_{\nu S} \setminus \nu S^1 = M \setminus \nu S, \partial) \xrightarrow{\bigcup \mathrm{id}_{\nu S}} \pi_0 \mathrm{Diff}(M, \partial) \\ \uparrow & & \uparrow \\ \pi_1(\mathrm{Emb}(\nu I_1, D^2 \setminus \nu I_2 = S^2 \times D^2), *) & \longrightarrow & \pi_0 \mathrm{Diff}(S^2 \times D^2 \natural S^2 \times D^2, \partial) \end{array}$$

The first line is just $\mathrm{ps}_{\nu S}$ so by Prop 3.3 it sends $[\nu T_R]$ to τ_W. On the other hand, when S^1 goes around T_R, it just goes along γ, winds R around and then comes back. This is exactly what I_1 does in the standard barbell. Thus we just implant $(I_1, S^2 \times D^2)$ into $M_{\nu S}$ (see Figures in Example 2.8 below). The resulting diffeomorphism in $M_{\nu S} \setminus \nu S^1 = M \setminus \nu S$ is the barbell diffeomorphism with implanted $\beta = (R_0, \text{normal sphere of } S^1 = S, \gamma)$, when extending to M it's still the same barbell diffeomorphism. □

Combining the above two propositions we get:

Corollary 2.7. *For any implanted $\beta = (R, S, \gamma)$ with S unknotted in M, the implanted barbell diffeomorphism is pseudo-isotopic to identity by $g_\beta \in \pi_0\mathcal{P}$ with the corresponding Cerf diagram being a single eye of (1,2)-handle pair. Moreover, it's induced from an element $[\nu T_R] \in \pi_1(\mathrm{Emb}(S^1 \times D^3, M\#S^1 \times S^3), *)$.*

Example 2.8. Here we draw the corresponding loop in $\pi_1(\mathrm{Emb}(S^1, S^1 \times D^3\#S^1 \times S^3), *)$ (for simplicity I omit the framing but it's just the natural framing induced by $\nu T_R|_{S^1}$ and $\nu(S^1, T_R)$) which corresponds to Gabai's constructions δ_k (see Figures 2.1 to 2.3).

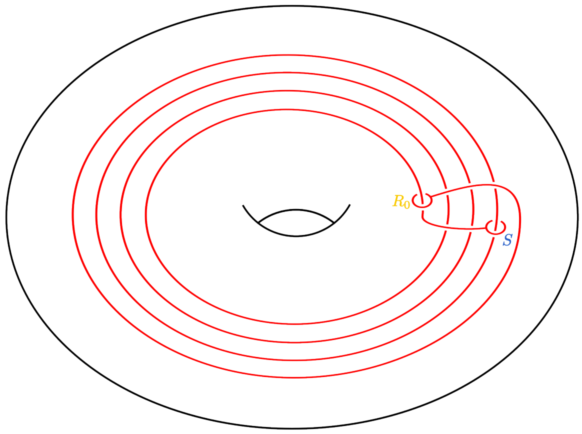

Figure 2.1 / The implanted barbell δ_4 in $M = S^1 \times D^3$

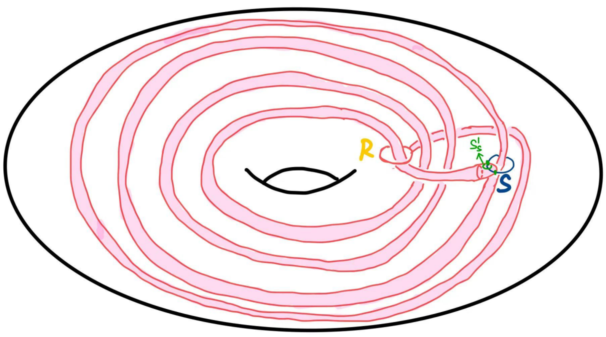

Figure 2.2 / Finger-push R_0 to get R such that $R \cap S = 2$ points

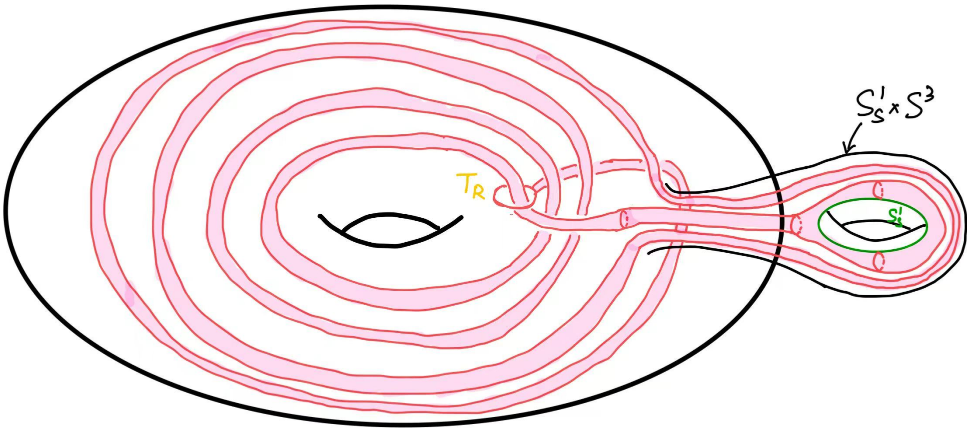

Figure 2.3 / Do surgery along S to get an embedded torus T_R in $S^1 \times D^3 \# S^1 \times S^3$ with the standard S^1 (green one), this T_R determines an element in $\pi_1(\mathrm{Emb}(S^1, S^1 \times D^3 \# S^1 \times S^3), *)$ which is a loop of handle decompositions $\mathcal{H} = \{H_t, t \in S^1\}$ (a loop of (1,2)-handle pair) of $S^1 \times D^3 \times I$ which results in the barbell diffeomorphism $\delta_4 = \tau_W$ where $W = (R, S)$.

3 Changing from (1,2)-handle pair to (2,3)-handle pair

The author first heard the excellent idea from [6]. This is just the 5-dimensional Kirby calculus, where in dimension 4, we use dotted unknotted $S^1 = \partial D^2 \subset M^3$ to represent a 1-handle, and framed $S^1 \subset M$ to represent 2-handle. The central idea is that when the framed S^1 is unknotted, *swapping the dotted circle and the 0-framed circle* changes the cobordism but results in the same boundary manifold.

In this chapter we first describe the dotted version of trivial 1 handles and trivial 2 handles in a 5-dimensional handle construction H on $X^5 = M \times I$, then derive the strategy of *dotted and 0-framed replacement*. After that, we move on to the one-parameter versions of both, namely, dotted version of a Cerf diagram containing a single eye of (1,2)-handle pair which corresponds to an element in $\pi_1(\mathrm{Emb}(S^1 \times D^3, M\#S^1 \times S^3), \mathrm{Emb}_0(S^1 \times D^3, M\#S^1 \times S^3))$, and carefully develop the *one-parameter version of dotted and 0-framed replacement.*

Consider the trivial cobordism $Z = M\times I$ from M to M, if we attach a one handle on top of Z, we get $Z_1 = Z \cup h_1$ which is a cobordism from M to $M_1 = (M \setminus S^0 \times D^4) \cup D^1 \times S^3 = M\#S^1 \times S^3$. If we push the one handle into original $Z = M \times I$, we know that it's the same as cutting a neighborhood of an unknotted properly embedded D^3 in $M \times I$, that is, $Z_1 = M \times I \setminus \nu D^3_{in}$ where D^3_{in} is obtained as follows: Choose an embedding $i : D^3 \to M \times I$, fix $i(\partial D^3)$ then push $i(D^3)$ into $M \times I$ to get D^3_{in}, then remove νD^3_{in} we get the same cobordism Z_1. Then when the attaching sphere S^1 of a two handle goes through $S^1 \times S^3 \setminus D^4 \subset M_1$, by slightly perturbing the attaching region, S^1 can be made disjoint from $S^1 \times * \in S^1 \times S^3$, then in the dotted version, the S^1 can be seen in $M \setminus \nu(\partial D^3)$. And whenever S^1 goes through the belt sphere $* \times S^3$, in the dotted version, $S^1 \subset M \setminus \nu(\partial D^3)$ intersects with that D^3. In short, the dotted version of a one handle can be described as:

Definition 3.1. For a 5-dimensional cobordism Z from $\partial_- Z$ to $\partial_+ Z = M$ with M connected, the *dotted version of describing a new cobordism* $Z_1 = Z \cup h_1$ is by choosing an embedding $\beta^\bullet : D^3 \hookrightarrow M$ with $\beta := \partial D^3$ denoting a dotted 2-sphere, then Z_1 is diffeomorphic to $Z \setminus \nu\beta^\bullet_{in}$ with $\beta^\bullet_{in} : D^3 \hookrightarrow Z$ obtained by pushing interior of $\beta^\bullet(D^3)$ into Z a bit. Whenever the attaching sphere of any 2-handle runs over the belt sphere of h_1, it intersects $\beta^\bullet$ (or equivalently, it links with β).

Running the same story of a trivial 2-handle, i.e. the attaching region is a 0-framed unknotted $S^1 = \partial D^2$ with $D^2 \hookrightarrow M$, we get the dotted version of a trivial 2-handle:

Definition 3.2. For a 5-dimensional cobordism Z from $\partial_- Z$ to $\partial_+ Z = M$ with M connected, the *dotted version of describing a new cobordism $Z_1 = Z \cup h_2$ with h_2 a trivial 2-handle*, is by choosing an embedding $\gamma^\bullet : D^2 \hookrightarrow M$ with $\gamma := \partial D^2$ denotes as a dotted 1-sphere, then Z_1 is diffeomorphic to $Z \setminus \nu\gamma^\bullet_{in}$ with $\gamma^\bullet_{in} : D^2 \hookrightarrow Z$ obtained by pushing interior of $\gamma^\bullet(D^2)$ into Z a bit. Whenever the attaching sphere of any 3-handle runs over the belt sphere of h_2, it intersects with $\gamma^\bullet$ (or equivalently, it links with γ).

Now if we are giving *both* data from the above 2 definitions, we can perform the *dotted and 0-framed replacement*:

Proposition 3.3. *For 5-dimensional cobordism Z with $\partial_+ Z = M$ Suppose we are given $\beta^\bullet : D^3 \hookrightarrow M$, $\gamma^\bullet : D^2 \hookrightarrow M$ with β, γ the corresponding boundary embeddings. If $\gamma S^1 \cap \beta S^2 = \emptyset$, then $\partial_+(Z \setminus \nu\beta^\bullet_{in} \cup_{(\gamma,0)} h_2) = \partial_+(Z \setminus \nu\gamma^\bullet_{in} \cup_{(\beta,0)} h_3)$. Here $(\gamma, 0)$ means that the attaching sphere of h_2 is γ and the framing is induced by $\nu(\gamma, \gamma^\bullet) \oplus \nu(\gamma^\bullet, M)|_\gamma$, similar for $(\beta, 0)$.*

Proof. Both sides are $(M \setminus (\nu\beta \cup \nu\gamma)) \cup_{(\beta,0)} D^3 \times S^1 \cup_{(\gamma,0)} D^2 \times S^2$. □

Remark 3.4. Later on in the dotted version, when we say the attaching region $S^{i-1} \times D^{5-i}$ of some handle h_i ($i = 2$ or 3) is *0-framed*, we will specify a $D^i \hookrightarrow M$ with the attaching sphere $S^{i-1} = \partial D^i$, and the framing is naturally induced by $\nu(S^{i-1}, D^i) \oplus \nu(D^i, M)|_{S^{i-1}}$. Usually the D^i is clear from the context. ⌟

Remark 3.5. Let $Z' = Z \setminus \nu\beta^\bullet_{in} \cup_{(\gamma,0)} h_2$ and $Z'' = Z \setminus \nu\gamma^\bullet_{in} \cup_{(\beta,0)} h_3$. Note that the two cobordisms result in the same boundary manifolds but they are different cobordisms themselves. In fact, when $Z = M \times I$, we can embed Z' into a bigger $M \times I$ such that $Z' \cup_{\partial_+ Z' = \partial_- \bar{Z}''} \bar{Z}'' = M \times I$ where $\bar{Z}''$ denotes the orientation-reversing cobordism from $\partial_+ Z''$ to $\partial_- Z''$ (see Figure 3.1).

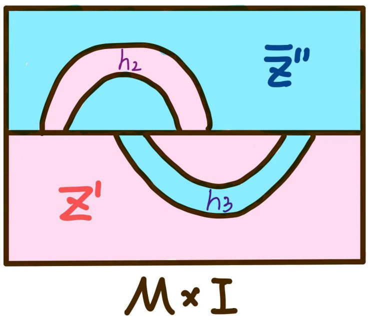

Figure 3.1 / $Z' \cup \bar{Z}'' = M \times I$

⌟

Recall what we obtained in the last chapter: For any half-unknotted barbell in M^4, we find a loop of framed S^1, i.e. an element in $\pi_1(\mathrm{Emb}(S^1 \times D^3, M \# S^1 \times S^3), *)$ which represents (by the map $\mathrm{ps}_{\nu S}$) that barbell diffeomorphism. And thus, it gives a Cerf diagram containing a single eye of (1,2)-handle pair which results in the barbell diffeomorphism. And our original goal is to find a Cerf diagram containing a single eye of (2,3)-handle pair resulting in the same barbell diffeomorphism, then calculate its Hatcher-Wagoner invariants. Therefore, the only thing we need to do is to change the eye of (1,2)-handle pair into the eye of (2,3)-handle pair. Then the natural idea is to do the *one-parameter version of dotted and 0-framed replacement.* To do that, we need the following lemma(this appeared in the work of Gay in [6] when $M = S^4$, and there Gay was going from the opposite direction, namely, to change a certain eye of (2,3)-handle pair into an eye of (1,2)-handle pair, but the idea of the proof is totally the same):

Lemma 3.6. *Consider a one-parameter family of embeddings: $\beta_t^\bullet : D^3 \hookrightarrow M$ (denote $\beta_t = \partial\beta_t^\bullet$) and $\gamma_t : S^1 \hookrightarrow M, t \in [0,1]$ with the following properties:*

(1) *γ_0 extends to $\gamma_0^\bullet : D^2 \hookrightarrow M$, i.e. $\gamma_0^\bullet|_{\partial D^2} = \gamma_0$, moreover, $\gamma_0^\bullet$ intersects β_0 transversely at a single point.*

(2) *$\beta_1^\bullet = \beta_0^\bullet$ and $\gamma_1 = \gamma_0$.*

(3) *For all $t \in [0,1]$, $\gamma_t \cap \beta_t = \emptyset$.*

(4) *γ_i intersects $\beta_i^\bullet$ transversely at a single point, for $i \in \{0,1\}$.*

Then there exists an extension of $\beta_t^\bullet$ and γ_t to $t \in [0,3]$ and a one-parameter family of embeddings $\gamma_t^\bullet : D^2 \hookrightarrow M, t \in [0,3]$ satisfying:

(1) *For all $t \in [0,1]$, γ_t and $\beta_t^\bullet$ stay the same as given.*

(2) *For $t \in [0,3]$, $\partial\gamma_t^\bullet = \gamma_t$.*

(3) *$\beta_3^\bullet = \beta_0^\bullet$ and $\gamma_3^\bullet = \gamma_0^\bullet$.*

(4) *For all $t \in [1,3]$, γ_t intersects $\beta_t^\bullet$ transversely at a single point.*

(5) *$\beta_0 = \beta_3$ intersects $\gamma_0^\bullet = \gamma_3^\bullet$ transversely at a single point.*

(6) *The path $\gamma_t^\bullet, t \in [0,3]$ is null homotopic rel $t \in \{0,3\}$ in $\mathrm{Emb}(D^2, M)$.*

Proof. First we construct $\gamma_t^\bullet$ for $t \in [0,1]$: Since $\beta_t \cap \gamma_t = \emptyset$, choose an isotopy $\psi_t : M \to M$ which sends γ_0 to γ_t and sends β_0 to β_t, then let $\gamma_t^\bullet := \psi_t(\gamma_0^\bullet)$.

But now $\gamma_1^\bullet$ may not be equal to $\gamma_0^\bullet$. Then we construct $\gamma_t^\bullet$ and $\beta_t^\bullet$ for $t \in [1,2]$ such that $\gamma_0^\bullet = \gamma_2^\bullet$: Let $\phi_t : M \to M$ be the isotopy such that $\phi_t(\gamma_1^\bullet) = \gamma_{2-t}^\bullet$ then define $\gamma_t^\bullet = \phi_t(\gamma_1^\bullet)$ and $\beta_t^\bullet = \phi_t(\beta_1^\bullet)$.

But now $\beta_2^\bullet$ may not be equal to $\beta_0^\bullet$. Then we construct $\gamma_t^\bullet$ and $\beta_t^\bullet$ for $t \in [2,3]$ such that $\gamma_t^\bullet = \gamma_2^\bullet = \gamma_0^\bullet$ for $t \in [2,3]$ and $\beta_3^\bullet = \beta_0^\bullet$: By condition (4), $\beta_0^\bullet$ is a normal disk of γ_0 and $\beta_1^\bullet$ is a normal disk of γ_1, since $\gamma_2 = \phi_2(\gamma_1) = \gamma_0$ and $\beta_2^\bullet = \phi_2(\beta_1^\bullet)$, so $\beta_2^\bullet$ is also a normal disk of $\gamma_2 = \gamma_0$. That means, both $\beta_2^\bullet$ and $\beta_0^\bullet$ are normal disks of γ_0, then we can isotope $\beta_2^\bullet$ to $\beta_0^\bullet$ during $t \in [2,3]$, one way is to shrink $\beta_2^\bullet$ to $\beta_t^\bullet$ inside $\beta_2^\bullet$ to a standard normal ball of γ_0 and then expand back to $\beta_0^\bullet$.

Then from the construction one can see directly that conditions (1), (2), (3), (5) are satisfied. For condition (4), for $t \in [1,2], |\gamma_t \cap \beta_t^\bullet| = |\phi_t\gamma_1 \cap \phi_t\beta_1^\bullet| = |\gamma_1 \cap \beta_1^\bullet| = 1$, for $t \in [2,3]$, $\beta_t^\bullet$ is always a normal disk of $\gamma_t = \gamma_0$ so satisfied. For condition (6), by $\gamma_t^\bullet = \gamma_{2-t}^\bullet, t \in [0,2]$ and $\gamma_t^\bullet = \gamma_0^\bullet, t \in [2,3]$, then of course the condition is satisfied. □

Proposition 3.7 (one-parameter version of dotted and 0-framed replacement). *Suppose that $\beta_t^\bullet : D^3 \hookrightarrow M$, $\gamma_t : S^1 \hookrightarrow M$, $t \in [0,1]$ satisfying the four conditions in the above lemma, moreover, $\beta_t^\bullet = \beta_0^\bullet$, and $\gamma_0^\bullet$ is a standard normal disk of β_0, by using the above lemma, we extend $\beta_t^\bullet$ and γ_t to $t \in [0,3]$ and get $\gamma_t^\bullet$ with $\gamma_t = \partial\gamma_t^\bullet$ for $t \in [0,3]$. We further assume $\gamma_1^\bullet$ and $\gamma_0^\bullet$ yield the same framing for $\gamma_1 = \gamma_0$. Since $\beta_0^\bullet = \beta_3^\bullet$ and $\gamma_0^\bullet = \gamma_3^\bullet$,*

(1) *For $t \in [0,1]$, regard $\beta_t^\bullet = \beta_0^\bullet$ as dotted version of a trivial 1-handle, regard $\gamma_t = \partial\gamma_t^\bullet$ as a loop of 0-framed 2-handle. Then it gives $\mathcal{H}_1 \in \pi_0\mathcal{P}$ with Cerf diagram being a single eye of (1,2)-handle pair and results in $f_{\mathcal{H}_1} \in \pi_0 \operatorname{Diff}(M, \partial)$.*

(2) *For $t \in [0,3]$, regard $\gamma_t^\bullet$ as dotted version of a trivial 2-handle (the 2-handle can be made not moving for $t \in [0,3]$ by condition (6) in above lemma), regard $\beta_t = \partial\beta_t^\bullet$ as a loop of framed 3-handle. Then it gives $\mathcal{H}_2 \in \pi_0\mathcal{P}$ with Cerf diagram being a single eye of (2,3)-handle pair and results in $f_{\mathcal{H}_2} \in \pi_0 \operatorname{Diff}(M, \partial)$.*

Then $[f_{\mathcal{H}_1}] = [f_{\mathcal{H}_2}] \in \pi_0 \operatorname{Diff}(M, \partial)$.

Proof. First consider $\beta_t^\bullet, \gamma_t = \partial\gamma_t^\bullet, t \in [0,3]$, this loop of handle decompositions of $M \times I$ yields $\mathcal{H}_0 \in \pi_0\mathcal{P}$ with Cerf diagram being a single eye of (1,2)-handle pair. And by doing dotted and 0-framed replacement *pointwise* for $t \in [0,3]$ from $\mathcal{H}_0$, we get another pseudo-isotopy which is exactly $\mathcal{H}_2 \in \pi_0\mathcal{P}$, thus $f_{\mathcal{H}_0} = f_{\mathcal{H}_2}$.

But note that $\beta_1^\bullet = \beta_3^\bullet$ and $\gamma_1 = \gamma_3$ with the same framing induced by $\gamma_1^\bullet$ and $\gamma_3^\bullet$, so for $t \in [1,3]$, $\beta_t^\bullet$ and $\gamma_t = \partial\gamma_t^\bullet$ also determines a loop of handle constructions for $M \times I$, thus a pseudo-isotopy $\mathcal{H} \in \pi_0\mathcal{P}$. But here $\beta_t^\bullet$ intersects γ_t transversely at a single point for all $t \in [1,3]$, which means they are all at cancelling position for all t, thus $[\mathcal{H}] = [\mathrm{id}] \in \pi_0\mathcal{P}$. So $\beta_t^\bullet, \gamma_t = \partial\gamma_t^\bullet, t \in [0,3]$ together gives a pseudo-isotopy $[\mathcal{H}_0 = \mathcal{H}_1 * \mathcal{H}] = [\mathcal{H}_1] \in \pi_0\mathcal{P}$.

Thus $[f_{\mathcal{H}_1}] = [f_{\mathcal{H}_0}] = [f_{\mathcal{H}_2}] \in \pi_0 \operatorname{Diff}(M, \partial)$. □

For any half-unknotted implanted barbell $\beta = (R, S, \gamma)$, from the last chapter, we get a loop of (1,2)-handle pair $g_\beta \in \pi_0\mathcal{P}$ resulting in the barbell diffeomorphism. By first changing to the dotted version and then applying Proposition 3.7 , we can see that there is a natural extension $\gamma_t^\bullet, t \in [0,1]$ by pulling $\gamma_0^\bullet$ along γ_t in T_R, i.e. $\gamma_t^\bullet = \gamma_0^\bullet \cup_{s\in[0,t]} \gamma_s$. Thus $\gamma_1^\bullet = \gamma_0^\bullet$ $\text{tube}_\gamma R$. Thus framing of $\gamma_1 = \gamma_0$ induced by $\gamma_1^\bullet$ is the same as that induced by $\gamma_0^\bullet$ (For moving pictures of $\gamma_t^\bullet$ in the case of $\beta = \delta_k$, see Figure 3.4). Thus we have:

Corollary 3.8. *For any half-unknotted implanted barbell $\beta = (R, S, \gamma)$ in M^4, there exists $f_\beta \in \pi_0\mathcal{P}$ whose Cerf diagram contains only a single eye of (2,3)-handle pair resulting in that barbell diffeomorphism. In particular, the first Hatcher-Wagoner invariant $\Sigma(f_\beta) = 0 \in \mathrm{Wh}_2(\pi_1 M)$.*

Remark 3.9. We are not going into the definition of the first Hatcher-Wagoner invariant $\Sigma : \pi_0\mathcal{P} \to \mathrm{Wh}_2(\pi_1 M)$(to see a clear definition, read [3]), but briefly speaking, it records the handle slides that happen during $\{f_t : M \times I \to I\} \in \pi_1(\mathcal{F}, \mathcal{E})$. So when there's only a single eye of (2,3)-handle pair, there's at most one handle for each index, so no handle slides at all time. ⌟

Remark 3.10. Note that in the proof of the above proposition, we obtain

$$\mathcal{H}_2 = \{H_{2,t}, t \in [0,3] \text{ which is a loop of (2,3)-handle constructions of } M \times I\}$$

by doing dotted and 0-framed replacement *pointwise* from

$$\mathcal{H}_0 = \{H_{0,t}, t \in S^1 \text{ which is a loop of (1,2)-handle construction of } M \times I\}.$$

Denote $Z_{\mathcal{H}_0}$ (resp. $Z_{\mathcal{H}_2}$) as the corresponding cobordism from $S^1 \times M$ to $Y_{\mathcal{H}} = M \times I/((x,0) \sim (f_{\mathcal{H}_0}(x), 1))$ (resp. $Y_{\mathcal{H}_2} = M \times I/((x,0) \sim (f_{\mathcal{H}_2}(x),1)))$. Since $f_{\mathcal{H}_0} = f_{\mathcal{H}_2}$, by Remark 3.5 we know that $Z_{\mathcal{H}_0} \cup \bar{Z}_{\mathcal{H}_2} = S^1 \times X \times I$, thus $0 = \Sigma(\mathcal{H}_0) + \Sigma(\bar{\mathcal{H}}_2) = \Sigma(\mathcal{H}_1) + \Sigma(\bar{\mathcal{H}}_2)$. Similarly, $\Theta(\mathcal{H}_1) + \Theta(\bar{\mathcal{H}}_2) = 0$. By [3, section 9.1] we know that there exists an involution $\bar{\cdot}$ on both $\mathrm{Wh}_2(\pi_1 M)$ and $\mathrm{Wh}_1(\pi_1 M; \mathbb{Z}_2 \times \pi_2 M)$ such that $\Sigma(\bar{f}) = \bar{\Sigma}(f)$, $\Theta(\bar{f}) = \bar{\Theta}(f), \forall f \in \pi_0\mathcal{P}$. For implanted barbell β, we thus have $\Sigma(g_\beta) = -\bar{\Sigma}(f_\beta) = 0$, $\Theta(g_\beta) = -\bar{\Theta}(f_\beta)$. ⌟

Example 3.11. We apply the above procedure to Budney and Gabai's barbell diffeomorphisms δ_k, changing its corresponding loop of (1,2)-handle pair which we obtained in the last chapter into the loop of (2,3)-handle pair.

Recall the loop of (1,2)-handle constructions for Cerf diagram resulting in δ_k (again we let $k = 4$ for simplicity), as shown in Figure 3.2.

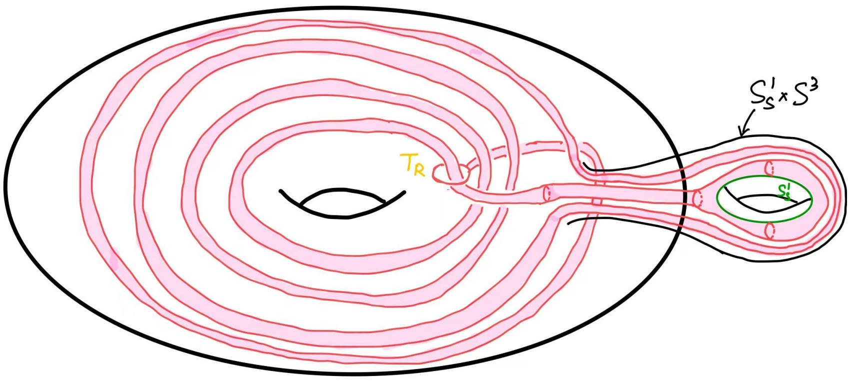

Figure 3.2 / Element in $\pi_1(\mathrm{Emb}(S^1 \times D^3, S^1 \times D^3 \# S^1 \times S^3), *)$ which represents barbell δ_k.

Change it to the one-parameter dotted version, i.e. $\beta_t^\bullet = \beta_0^\bullet : D^3 \hookrightarrow S^1 \times D^3$, $\gamma_t : S^1 \hookrightarrow S^1 \times D^3$, $\gamma_0 = \gamma_1$ a standard meridian of $\beta_0 = \partial\beta_0^\bullet$, $t \in [0,1]$ (see Figure 3.3).

Now following the constructions we made in Lemma 3.6, we need to find $\gamma_t^\bullet, t \in [0,1]$. Let $\gamma_0^\bullet$ just be the standard normal disk of $\partial\beta_0^\bullet$, then $\gamma_t^\bullet$ is just dragging $\gamma_0^\bullet$ onto T_R along the trajectory of S^1 in T_R, finally $\gamma_1^\bullet = \gamma_0^\bullet \#_{tube} R$ (see Figure 3.4).

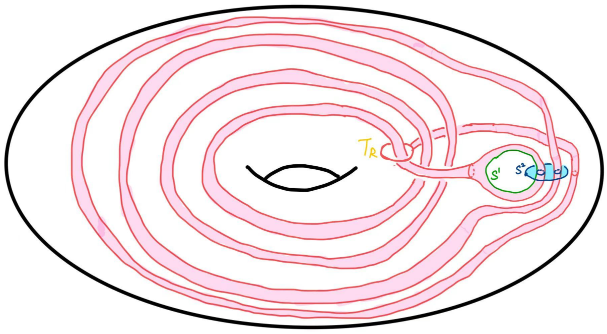

Figure 3.3 / Dotted version of the loop of (1,2)-handle pair. The dark blue is the dotted $S^2 = \partial\beta_0^\bullet$, the light blue region is $\beta_t^\bullet = \beta_0^\bullet : D^3 \hookrightarrow S^1 \times D^3$ which represents the 1-handle.

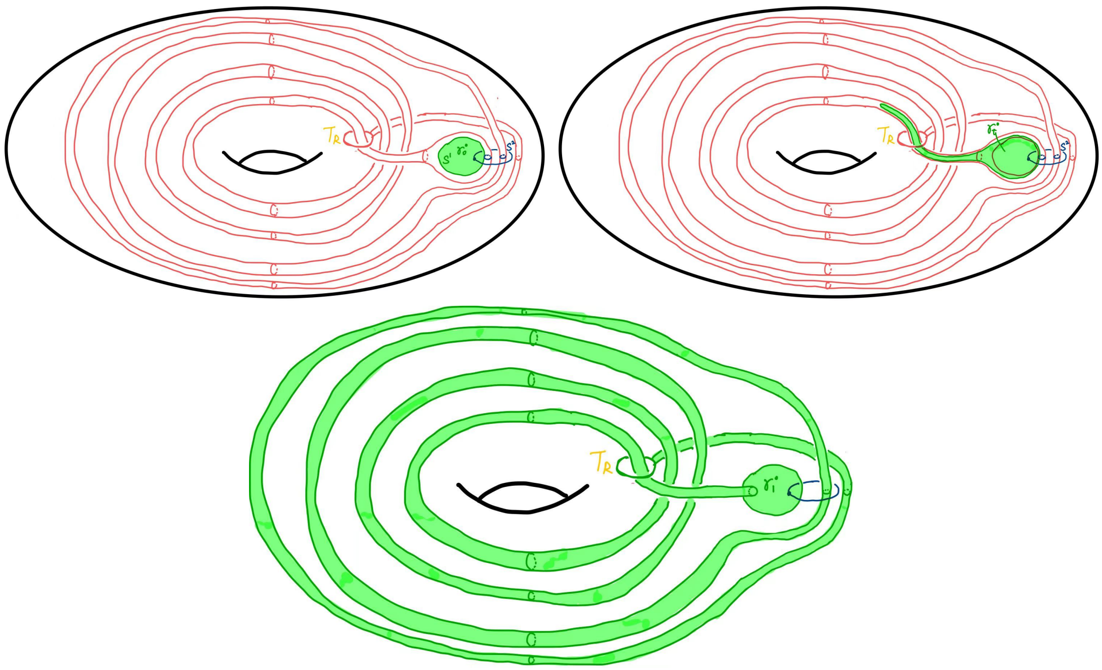

Figure 3.4 / Movement of $\gamma_t^\bullet : D^2 \hookrightarrow S^1 \times D^3$. The figures draw $\gamma_0^\bullet, \gamma_t^\bullet, \gamma_1^\bullet$.

Now at $t \in [1,2]$, as the constructions in Lemma 3.6 read, consider $\phi_t : S^1 \times D^3 \to S^1 \times D^3$, s.t. $\gamma_t^\bullet = \phi_t(\gamma_1^\bullet) = \gamma_{2-t}^\bullet$, $\beta_t^\bullet = \phi_t(\beta_1^\bullet = \beta_0^\bullet)$. In the interval $t \in [1,2]$, $\beta_t = \beta_0$ intersects $\gamma_t^\bullet$ at a single point. While $\beta_2^\bullet$ intersects $\gamma_2^\bullet$ at $S^1 \cup I^1$, so when $t \in [2,3]$, β_t shrinks in $\beta_2^\bullet$ to a standard normal sphere of $\gamma_2 = \gamma_0$ so that $\beta_3 = \beta_0$. Therefore, when during $t \in [2,3]$, β_t will intersect $\gamma_t^\bullet = \gamma_0^\bullet$ at 1 point, then at 3 points, then 1 point, which will become essential to our calculations of Hatcher-Wagoner invariants. So in order to understand how $\beta_t, t \in [2,3]$ shrinks, we only need to understand $\beta_2^\bullet$:

$\beta_2^\bullet$ is the isotopy image of $\beta_1^\bullet = \beta_0^\bullet$ when pulling γ_1 back to standard γ_0 along R, but this is just the *backward barbell diffeomorphism* with data $\beta = (R_0, S, \gamma)$. What's more, the $\beta_1^\bullet$ is just a mid-ball of this implanted barbell (see Figure 3.5)! Thus in [4] we know the embedded surgery description of the mid-ball after performing a barbell. In Figure 3.6 we draw the embedded surgery explicitly to get $\beta_2^\bullet$.

Then performing *one-parameter version of dotted and 0-framed replacement* we regard $\gamma_t^\bullet, t \in [0,3]$ as the dotted version of $h_{2,t}, t \in [0,3]$ and regard $\beta_t, t \in [0,3]$ as the attaching spheres of $h_{3,t}, t \in [0,3]$. Thus we change the Cerf diagram of a (1,2)-handle pair to the Cerf diagram of a (2,3)-handle pair, both resulting in barbell δ_k.

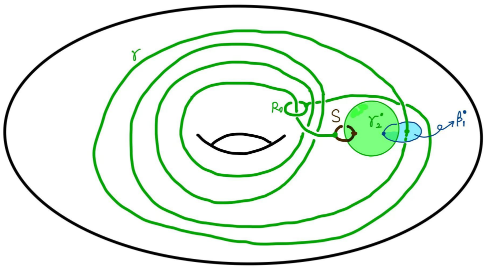

Figure 3.5 / Pulling γ_1 back to γ_0(therefore pulling $\gamma_1^\bullet$ back to $\gamma_0^\bullet$) is equivalent to doing the barbell diffeomorphism $\beta = (R_0, S, \gamma)$.

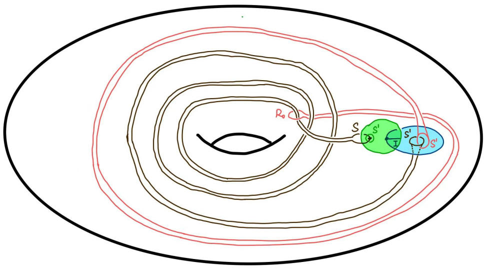

Figure 3.6 / The mid-ball $\beta_1^\bullet$ becomes $\beta_2^\bullet = \partial(\beta_1^\bullet \cup h_{2,b} \cup h_{2,r})$ where $h_{2,b}$ is the 2-handle with attaching S_b^1 the brown one and core $D_b^2 = \{S \text{ tube along the brown arc}\}$, $h_{2,r}$ is the 2-handle with attaching S_r^1 the red one and core $D_r^2 = \{R \text{ tube along the red arc}\}$. As a result, $\beta_2^\bullet = (\beta_1^\bullet \setminus (S_b^1 \times D^2 \cup S_r^1 \times D^2)) \cup (D_b^2 \times S^1 \cup D_r^2 \times S^1)$. Therefore, $\beta_2^\bullet \cap \gamma_2^\bullet =$ the green S^1 and the blue I in the picture.

4 Computation for the second Hatcher-Wagoner invariant

In this chapter, first we recall the definition for the second Hatcher-Wagoner invariant, namely, $\Theta : \ker \Sigma \to \mathrm{Wh}_1(\pi_1 M, \mathbb{Z}_2 \times \pi_2 M)/\chi(K_3\mathbb{Z}[\pi_1 M])$. For simplicity, we restrict to the case when the Cerf diagram contains only a single eye of (2,3)-handle pair. For general definitions and well-definedness, see [1] and [3].

Recall that $\mathrm{Wh}_1(\pi_1 M, \mathbb{Z}_2 \times \pi_2 M) = (\mathbb{Z}_2 \times \pi_2 M)[\pi_1 M]/(\beta \cdot [1], \alpha \cdot [\sigma] - \alpha^\tau \cdot [\tau\sigma\tau^{-1}], \alpha, \beta \in \mathbb{Z}_2 \times \pi_2 M, \tau, \sigma \in \pi_1 M)$ (see [3] for details).

To define Θ, we need the following data: For $\{f_t, t \in I\} \in \pi_1(\mathcal{F}, \mathcal{E})$ with Cerf diagram only a single eye of (2,3)-handle pair, suppose that the eye appears just before $t = \epsilon$ at critical value $\frac{1}{2}$, and ends right after $t = 1-\epsilon$ at the same critical value, then consider $V := \bigcup_{t\in[\epsilon,1-\epsilon]} f_t^{-1}(1/2) = M' \times [\epsilon, 1-\epsilon]$ where $M' = M\#S^2 \times S^2$. For each $f_t, t \in [\epsilon, 1-\epsilon]$, we call the belt 2-sphere of the 2-handle B_t, the attaching 2-sphere of the 3-handle A_t. Let $A := \bigcup_{t\in[\epsilon,1-\epsilon]} A_t \cong S^2 \times I \subset V$, $B := \bigcup_{t\in[\epsilon,1-\epsilon]} B_t \cong S^2 \times I \subset V$. Without loss of generality, we may assume that all intersections among A, B, V are transverse. Then we know that $A \cap B = I \sqcup_{i=1,\dots,k} S^1$, we denote the i-th S^1 as S^1_i. Now, choose a base point $* \in X \times 0 \subset X \times I$, a fixed $*' \in I \subset A \cap B$ and a path $\delta_0 \subset X \times I : * \to *'$. For each i, choose a point $p_i \in S^1_i$, and two paths: $\delta_i^A \subset A$: $*' \to p_i$, $\delta_i^B \subset B : *' \to p_i$. *Denote* $\gamma_i := \delta_i^B * (\delta_i^A)^{-1} \in \pi_1(X \times I, *')$ and $\gamma_i^* := \gamma_i^{\delta_0} \in \pi_1(X \times I, *)$ by pulling γ_i from the base point $*'$ back to the base point $*$ along δ_0. Also note that both A and B are simply connected, then choose $D_i^A \looparrowright A$ such that $\partial D_i^A = S^1_i$, $D_i^B \looparrowright B$ such that $\partial D_i^B = S^1_i$, so *denote* $\beta_i := [D_i^A \cup D_i^B] \in \pi_2(X \times I, p_i)$ and $\beta_i^* := \beta_i^{\delta_0 * \delta_i^B} \in \pi_2(X \times I, *)$ by pulling β_i from base point p_1 back to base point $*$ along $\delta_0 * \delta_i^B$.

To define the $\mathbb{Z}_2$ component, we consider two framings of the same $\mathbb{R}^2$-vector bundle on S^1_i. Consider $\nu(S^1_i, B)$(the normal bundle of S^1_i in B), since A intersects B transversely in V, then $E = \nu(S^1_i, B) = \nu(A, V)|_{S^1_i}$ is a 2-dimensional trivial vector bundle over S^1_i. We define two framings on E: $e_{i,B}$ is the framing naturally induced by $D_i^B \looparrowright B$, $e_{i,A}$ is induced by the attaching data for the 3-handle, namely, $A_t \times D^2 \hookrightarrow M' = M'_t$. Then $e_{i,A} - e_{i,B} \in \pi_1(SO_2) = \mathbb{Z}$, we define $s_i := e_{i,A} - e_{i,B}$ mod 2. Thus for every i, we have defined $s_i \in \mathbb{Z}_2$.

Using all those notations above, we can define the second Hatcher-Wagoner invariant Θ for $F := \{f_t, t \in I\} \in \pi_1(\mathcal{F}, \mathcal{E})$:

Definition 4.1. For $F := \{f_t, t \in I\} \in \pi_1(\mathcal{F}, \mathcal{E})$ with Cerf diagram only a single eye of (2,3)-handle pair, define $\Theta(F) := \sum_{i=1,\dots,k}(s_i, \beta_i^*) \cdot [\gamma_i^*]$.

Example 4.2. Now we calculate the second Hatcher-Wagoner invariant for implanted barbell diffeomorphism δ_k:

Using results in last chapter, by projecting the movements to a single $S^1 \times D^3 \# S^1 \times S^3$, and then using the dotted version and *one-parameter version of dotted and 0-framed replacement* in $S^1 \times D^3$, we use β_t and $\gamma_t^\bullet$ in $S^1 \times D^3$ to represent A_t, B_t in $S^1 \times D^3 \# S^1 \times S^3$. We see that the attaching sphere $A_t = \beta_t$ intersects $B_t = \gamma_t^\bullet \cup D^2$ at a single point for $t \in [0,2]$ and $\bigcup_{t\in[2,3]} A_t \cap B_t = \beta_2^\bullet \cap \gamma_2^\bullet = I \sqcup S^1$ with I the blue arc in Figure 3.6, and S^1 the green one in Figure 3.6. Thus from Figure 3.6 we see $\gamma^* = \gamma_1^* = (k-1)^* = k-1 \in \pi_1(S^1 \times D^3, *) = \mathbb{Z}$.

To calculate β^*, we need to find $D^B \hookrightarrow \gamma_2^\bullet \times I$ and $D^A \hookrightarrow \beta_2^\bullet$ which bound $\partial D^A = \partial D^B = S^1$. By projecting to a single $S^1 \times D^3$ for $t \in [2,3]$, we know that $\gamma_t^\bullet = \gamma_2^\bullet$ is the standard green one in Figure 3.6 which won't change, so it's enough to find $D^B \hookrightarrow \gamma_2^\bullet$. For D^B, we have an immediate choice in Figure 3.6, which is just the small disk in green $\gamma_2^\bullet$ which bounds the green S^1. For D^A, we pull the green S^1, which is the meridian of the brown tube, along the black tube back to the meridian of brown $S^1 \subset$ the blue D^3, which is just a parallel of red $S^1 \subset$ the blue D^3, then capping with the red D^2 which is the red long tube connected with R_0. In all, $\beta^* = \beta_1^* = (R_0^\gamma)^* \in \pi_2(S^1 \times D^3, *) = 0$, here γ is the arc in the definition of barbell, which is the arc connecting S with R_0.

Now we calculate $s = s_1$. For $s_1^A =$ attaching framing of $\nu(A,V)|_{t\in[2,3],S^1\subset A}$, from the dotted version we know that $\{A_t | t \in [2,3]\} \subset \beta_2^\bullet$ is 0-framed, thus s_1^A has a chapter which is the normal bundle of $\beta_2^\bullet$ in $S^1 \times D^3$, which is exactly a normal chapter of $\nu(S^1, \gamma_2^\bullet) \subset \nu(S^1, \gamma_2^\bullet \times I = B|_{t\in[2,3]})$ when restricted to the green S^1. Thus $s^A = s^B$, $s = 0$.

Thus we obtain that $\Theta(\delta_k) = (0, (R_0^\gamma)^*)[k-1] = (0,0)[k-1] = 0 \in \mathrm{Wh}_1(\mathbb{Z}, \mathbb{Z}_2 \times 0)$.

The calculation can be generalized similarly to any half-unknotted implanted barbell $\beta = (R_0, S, \gamma)$ with S unknotted. After doing an isotopy of M we can put $S = \partial\beta_0^\bullet$ into the standard position and use strategies in Chapter 3 to find a Cerf diagram containing a single eye of (2,3)-handle pair with $\gamma_t^\bullet, \beta_t, t \in [0,3]$. Note that for $t \in [0,2]$, β_t intersects $\gamma_t^\bullet$ at a single point, so as the example suggests, the second Hatcher-Wagoner invariant can be calculated just using $\beta_2^\bullet$ and $\gamma_2^\bullet = \gamma_0^\bullet$. Assume we have obtained $\beta_2^\bullet \subset M$, $\beta_2^\bullet \cap \gamma_2^\bullet = I \sqcup_{i=1,\ldots,k} S^1$, then the same argument of the example works in general case, i.e. using notations above, we have

$$\Theta(F) = \sum_{i=1,\ldots,k} (s_i, \beta_i^*) \cdot [\gamma_i^*]$$

where for each i, $\gamma_i = \delta_i^B * (\delta_i^A)^{-1}$ with $\delta_i^A \hookrightarrow \beta_2^\bullet$, $\delta_i^B \hookrightarrow \gamma_2^\bullet$ connecting $*'$ and $p_i \in S_i^1$, $\beta_i = D_i^A \cup D_i^B$ with $D_i^A \hookrightarrow \beta_2^\bullet$ and $D_i^B \hookrightarrow \gamma_2^\bullet$, $s_i = e_{i,A} - e_{i,B} \bmod 2$ with $E = \nu(S_i^1, \gamma_2^\bullet \times I) = \nu(\beta_2^\bullet, M \times I)|_{S_i^1}$. $\nu(S_i^1, \gamma_2^\bullet \times I) = \nu(S_i^1, \gamma_2^\bullet) \oplus E' = \nu(S_i^1, D_i^B) \oplus E'$, $\nu(\beta_2^\bullet, M \times I)|_{S_i^1} = \nu(\beta_2^\bullet, M) \oplus E''$ where E', E'' are two 1-dimensional trivial bundle over S_i^1. $e_{i,B}$ is the framing induced by $\nu(S_i^1, \gamma_2^\bullet)$ and $e_{i,A}$ is the framing induced by $\nu(\beta_2^\bullet, M)|_{S_i^1}$. But since $\beta_2^\bullet$ intersects $\gamma_2^\bullet$ transversely at S_i^1 in M, so $\nu(S_i^1, \gamma_2^\bullet) = \nu(\beta_2^\bullet, M)|_{S_i^1}$, thus, $s_i = e_{i,A} - e_{i,B} = 0$.

To calculate β_i^* and γ_i^*, we need a smarter change: It can be very hard to visualize $\beta_2^\bullet$, especially when R_0 goes through $\beta_0^\bullet$ in a strange way ($R_0 \cap \beta_0^\bullet$ may be knotted in $D^3 = \beta_0^\bullet$).

But we know that there is an isotopy $\phi_2 : M \to M$ which is isotopic to identity such that $\phi_2(\beta_1^\bullet = \beta_0^\bullet) = \beta_2^\bullet$, $\phi_2(\gamma_1^\bullet) = \gamma_2^\bullet$. This means we can pullback every data we need to $\beta_1^\bullet$ and $\gamma_1^\bullet$. Then it will be easier to visualize in the given implanted barbell β.

Note that $\beta_1^\bullet = \beta_0^\bullet$ is the given unknotted ball which bounds S, i.e. $S = \partial\beta_1^\bullet$. Also recall that $\gamma_1^\bullet = \gamma_0^\bullet$ tube$_\gamma$ R_0. Without loss of generality, we may assume that $\beta_0^\bullet \cap R_0 = \sqcup_{i=1,\dots,k} S_i^1$, $\mathrm{int}(\beta_0^\bullet) \cap \gamma = \emptyset$ (by finger-pushing R_0 along γ without reaching S, such that the arc γ of the implanted barbell β can be made arbitrarily short). Let $*_0 = S \cap \gamma$ be the base point, note that $*_0$ will not move during the whole time $t \in [0,3]$(in Figure 3.6, $*_0$ is the left vertex of the blue I). For each i, find a path $\delta_i^B \subset \beta = (R_0, S, \gamma)$ which is a path from $*_0$ to $p_i \in S_i^1$. Also, for each i, S_i^1 divides R_0 into two embedded disk D_i and D_i', where D_i' is the one connected to the arc γ. Let $D_i^B = D_i$. Now comes the following main theorem:

Theorem 4.3. *For a half-unknotted implanted barbell $\beta = (R_0, S, \gamma)$ with $S = \partial\beta_0^\bullet$ where $\beta_0^\bullet : D^3 \hookrightarrow M$, by finger-pushing R_0 along the arc, we can make γ short enough such that int$(\beta_0^\bullet) \cap \gamma = \emptyset$. Now suppose that $\beta_0^\bullet \cap R_0 = \bigsqcup_{i=1}^k S_i^1$. Choose $p_i \in S_i^1$ and let $*_0 = \gamma \cap S$ be the base point. For each i, find a path $\delta_i^B \subset \beta = (R_0, S, \gamma)$ which is a path from $*_0$ to $p_i \in S_i^1$. Also, for each i, S_i^1 divides R_0 into two embedded disks D_i and D_i', where D_i' is the one connected to the arc γ. Let $D_i^B = D_i$. Then the $f_\beta \in \pi_0\mathcal{P}$ we constructed in Corollary 3.8 which results in the implanted barbell diffeomorphism w.r.t. β satisfies:*

$$\Theta(f_\beta) = \sum_{i=1,\dots,k} (0, [D_i^B]^{\delta_i^B}) \cdot [\delta_i^B]$$

Here we identify $\pi_i(M, *_0)$ with $\pi_i(M, \beta_0^\bullet)$ so that $[D_i] \in \pi_2(M, \beta_0^\bullet)$ and $\delta_i^B \in \pi_1(M, \beta_0^\bullet)$.

Proof. In the above discussion we already showed that $s_i = 0 \in \mathbb{Z}_2$ for all i, we only need to show that $[D_i]^{\delta_i^B} = \beta_i^*$ and $[\delta_i^B] = \gamma_i^*$. But since we choose $*_0 \in M$ which won't move during the whole procedure, then $* = (*_0, 0) \in M \times 0$, $*' = (*_0, 1/2) \in (M')_t = M \# S^2 \times S^2$. So for the calculation, we can choose β_i^* and γ_i^* based at $*_0 \in M$. Then by definition and by changing data to $\gamma_1^\bullet = \gamma_0^\bullet$ tube$_\gamma$ R_0 and $\beta_1^\bullet = \beta_0^\bullet$, we pick $\delta_i^A \subset \beta_1^\bullet$ connecting $*_0$ with p_i, pick $D_i^A \looparrowright \beta_1^\bullet$ with $\partial D_i^A = S_i^1$. Then by definition $\beta_i^* = [\phi_2((D_i^A \cup D_i^B)^{\delta_i^B})] = [(D_i^A \cup D_i^B)^{\delta_i^B}] \in \pi_2(M, *_0)$, since by Lemma 3.6, $\phi_t : M \to M, t \in [1,2]$ is a one-parameter family of isotopies of M with $\phi_1 = \mathrm{id}$ and $\phi_t(*_0) = *_0, \forall t \in [1,2]$. But $\beta_0^\bullet$ is an embedded contractible D^3, so $\beta_i^* = [D_i^B]^{\delta_i^B}$. Also for the same reason, $\gamma_i^* = \delta_i^B * (\delta_i^A)^{-1} \in \pi_1(M, *_0)$ which is $\delta_i^B \in \pi_1(M, \beta_0^\bullet)$. □

Example 4.4. Using Figures 3.3 and 3.4 (where we draw $\gamma_1^\bullet$ and $\beta_1^\bullet = \beta_0^\bullet$ explicitly when $\beta = \delta_k$ in $S^1 \times D^3$) and results in Theorem 4.3, one can easily see that for the implanted barbell δ_k in $S^1 \times D^3$, $\Theta(f_{\delta_k}) = (0, 0) \cdot [k-1] = 0$, which coincides with Example 4.2.

5 Generalizations to immersed barbell diffeomorphisms

In this chapter we generalize implanted barbell diffeomorphisms to *immersed barbell diffeomorphisms* from the perspective of graspers, then we show that half-unknotted immersed barbell diffeomorphisms are pseudo-isotopic to identity, and we can similarly find $f_\beta \in \pi_0\mathcal{P}$ resulting in that immersed barbell diffeomorphism such that its Cerf diagram contains a single eye of (2,3)-handle pair. Then we compute the Hatcher-Wagoner invariants for f_β. As a corollary, we show that for every $\sigma \in \pi_2 M$ and $\forall \gamma \in \pi_1 M$, $(0,\sigma)\cdot[\gamma] \in \mathrm{Wh}_1(\pi_1 M, \mathbb{Z}_2 \times \pi_2 M)$ can be realized by a half-unknotted immersed barbell diffeomorphism, i.e. there exists $f \in \pi_0\mathcal{P}$ resulting in a half-unknotted immersed barbell diffeomorphism such that $\Sigma(f) = 0, \Theta(f) = (0,\sigma)\cdot[\gamma]$.

Definition 5.1. An *immersed barbell* $\beta = (R_0, S, \gamma)$ consists of a framed immersed 2-sphere R_0 in M, a framed embedded 2-sphere S in M, and a framed embedded arc γ in M connecting R_0 and S. By finger-pushing R_0 along γ into S to get another framed immersed 2-sphere R in M such that $R \cap S = 2$ pts, we do surgery along νS to get $M_{\nu S}$, then after surgery we get $T_R = (R \setminus D^2 \times \partial I) \cup S^1 \times I$ which is a framed immersed torus. As in Chapter 2, νT_R also defines an element $[\nu T_R] \in \pi_1(\mathrm{Emb}(\nu S^1, M_{\nu S}), *)$. We define the corresponding *immersed barbell diffeomorphism* as $\mathrm{ps}_{\nu S}[\nu T_R]$, where we define $\mathrm{ps}_{\nu S}$ in Definition 2.2.

In the same manner of Proposition 2.4 we can show that, when S is unknotted, there is a loop $g_\beta \in \pi_0\mathcal{P}$ of handle constructions of $M \times I$ containing a single (1,2)-handle pair which results in that immersed barbell diffeomorphism. And by doing *one-parameter version of dotted and 0-framed replacement* stated in Lemma 3.6 we change the loop of (1,2)-handle pair g_β to a loop of (2,3)-handle pair $f_\beta \in \pi_0\mathcal{P}$, thus we can compute the Hatcher-Wagoner invariants for f_β.

Throughout the process, the only difference from the embedded case is that, to obtain $\gamma_t^\bullet, t \in [0,1]$ with $\gamma_t = \partial\gamma_t^\bullet$ which is the given loop of attaching spheres of the 2-handles, in the previous embedded case, we naturally choose $\gamma_t^\bullet = \gamma_0^\bullet \cup_{s\in[0,t]} \gamma_s$ (that is, pull $\gamma_0^\bullet$ along $\gamma_t, t \in [0,1]$), but now since T_R is an immersed torus, it may have double self-intersection points, then there'll be $t_0 < t_1$ such that $p := \gamma_{t_0} \cap \gamma_{t_1} \neq \emptyset$ (by a reparametrization of T_R, which won't change the resulting $g_\beta \in \pi_0\mathcal{P}$, one can assume that for any $t_0 \neq t_1$, there's at most one intersection point between γ_{t_0} and γ_{t_1}). In this case when the intersection point is reached for the second time at $t = t_1$, one must finger-push the small intersection disk $D^2 = \{\text{normal disk of } T_R \text{ at point } p\} \subset \gamma_t^\bullet, t \in (t_0, t_1)$ towards the trajectory of γ_t to get an embedded $\gamma_t^\bullet, t > t_1$. Figure 5.1 is an illustration.

After understanding that we can deduce the following theorem, which is a generalization of Theorem 4.3:

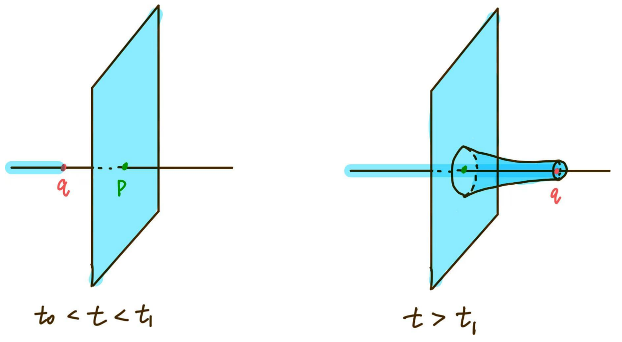

Figure 5.1 / Here we draw a local region of T_R near the self-intersection point in an $\mathbb{R}^3$-slice. $p = \gamma_{t_0} \cap \gamma_{t_1}$, $q \in \gamma_t$, and the blue region is the local part of $\gamma_t^\bullet$ in a $\mathbb{R}^3$-slice. When $t > t_1$, the blue disk $\subset \gamma_t^\bullet$ will be finger-pushed along γ_t to ensure $\gamma_t^\bullet$ is embedded.

Theorem 5.2. *For a half-unknotted immersed barbell $\beta = (R_0, S, \gamma)$ with $\beta_0^\bullet : D^3 \hookrightarrow M, S = \partial\beta_0^\bullet$, Perturb self-intersections of R_0 away from $\beta_0^\bullet$. By finger-pushing R_0 along the arc, we can make γ short enough such that int$(\beta_0^\bullet) \cap \gamma = \emptyset$. Now suppose that $\beta_0^\bullet \cap R_0 = \sqcup_{i=1,\dots,k} S_i^1$. Choose $p_i \in S_i^1$ and let $*_0 = \gamma \cap S$ be the base point. For each i, find a path $\delta_i^B \subset \beta = (R_0, S, \gamma)$ which is a path from $*_0$ to $p_i \in S_i^1$. Also, for each i, S_i^1 divides R_0 into two embedded disks D_i and D_i', where D_i' is the one connected to the arc γ. Let $D_i^B = D_i$. Then the $f_\beta \in \pi_0\mathcal{P}$ we constructed which results in the immersed barbell diffeomorphism w.r.t. β satisfies:*

$$\Theta(f_\beta) = \sum_{i=1,\dots,k} (0, [D_i^B]^{\delta_i^B}) \cdot [\delta_i^B]$$

Here we identify $\pi_i(M, *_0)$ with $\pi_i(M, \beta_0^\bullet)$ so that $[D_i] \in \pi_2(M, \beta_0^\bullet)$ and $\delta_i^B \in \pi_1(M, \beta_0^\bullet)$.

Proof. By finger-pushing R_0 along γ into S to get R, we then attach a 1-handle, in the dotted version, we push $\beta_0^\bullet$ into $M \times I$ and cut out a neighborhood of it. On the top we just do surgery on νS and get T_R which is an immersed framed torus in $M\#S^1 \times S^3$. On the left side of Figure 5.2 we draw the immersed T_R in the dotted version with standard $\gamma_0^\bullet$ in green. Thus we get initial data $\gamma_0^\bullet$, $\{\gamma_t, t \in [0,1]\} \in \pi_1(\text{Emb}(\nu S^1, M\#S^1 \times S^3), *)$ and $\beta_t^\bullet = \beta_0^\bullet, t \in [0,1]$. To use *one-parameter version of dotted and 0-framed replacement* and calculate $\Theta(f_\beta)$ we only need to find $\gamma_t^\bullet, t \in [0,1]$.

As in the process from the last chapter, we have $s_i = 0$, β_i^* and γ_i^* can be calculated in $\gamma_1^\bullet$ and $\beta_1^\bullet = \beta_0^\bullet$. By the discussions just above the theorem, we isotope $\gamma_0^\bullet$ along γ_t to $\gamma_t^\bullet$. Whenever we reach a self-intersection point $p = \gamma_{t_1}(v)$ ($\gamma_t : S^1 \to T_R \subset M : w \to \gamma_t(w)$) the second time, the normal disk at the intersection point needs to be stretched along $l_p := \cup_{t\in[t_1,1]}\gamma_t(v)$. Thus in the case T_R has just a single self-intersection point, $\gamma_1^\bullet = (\gamma_0^\bullet \text{ tube}_\gamma R_0 - D^2) \cup S^1 \times l_p \cup D_1^2$ where $S^1 \times l_p = SN(T_R)|_{l_p}$ (for X immersed in M, we use $SN(X)$ to denote the sphere bundle of $N(X)$ which is the normal bundle of X in M), that is, each S^1 is a meridian sphere, and $D_1^2 \subset S^2 = SN(\gamma_1)|_{\gamma_1(v)}$ which bounds $S^1 = SN(T_R)|_{\gamma_1(v)}$ such that $D_1^2 \cap \gamma_0^\bullet = \emptyset$ (see Figure 5.2 for an illustration).

When there are more double self-intersection points, by a nice reparametrization of T_R, we may assume the self-intersection points are $p_j = \gamma_{t_{0,j}}(w_j) \cap \gamma_{t_{1,j}}(v_j), t_{0,j} < t_{1,j}, j = 1, \dots, n$ and $\{v_j\}$ are distinct from each other. Then let $l_{p_j} := \cup_{t\in[t_{1,j},1]}\gamma_t(v_j)$. Then $\gamma_1^\bullet = (\gamma_0^\bullet \text{ tube}_\gamma R_0 -$

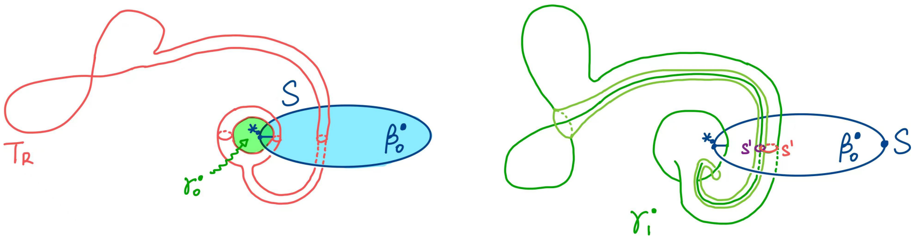

Figure 5.2 / Here we illustrate immersed torus T_R and the standard $\gamma_0^\bullet$ on the left and the desired $\gamma_1^\bullet$ on the right. If T_R is embedded, $\gamma_1^\bullet \cap \beta_0^\bullet = I \cap S^1$ where I is in blue and S^1 is in red. But since T_R has a self-intersection point, by finger-pushing a small disk near the intersection point along γ_t as we said in Figure 5.1, this induces another $S^1 \subset \gamma_1^\bullet \cap \beta_0^\bullet$ in purple, which is exactly a meridian of the red S^1.

$\sqcup_{j=1,\dots,n} D^2) \cup_{j=1,\dots,n} (S^1 \times l_{p_j}) \cup_{j=1,\dots,n} D^2_{1,n}$ where $S^1 \times l_{p_j} = SN(T_R)|_{l_{p,j}}$ and $D^2_{1,j} \subset S^2 = SN(\gamma_1)|_{\gamma_1(v_j)}$ which bounds $S^1 = SN(T_R)|_{\gamma_1(v_j)}$ such that $D^2_j \cap \gamma_0^\bullet = \emptyset$.

Then consider $\gamma_1^\bullet \cap \beta_1^\bullet$, it's $R \cap \beta_1^\bullet$ which are k disjoint S^1, namely $S^1_i, i = 1, \dots, k$, and several meridians of S^1_i in $\beta_0^\bullet$. But for every meridian $S^1 = SN(T_R)|_{x_m}, x_m = \gamma_{t_{j,m}}(v_j) \in l_{p_j}, t_{j,m} \in [t_{1,j}, 1]$ for some j (like the one in purple in Figure 5.2), $D^B = (SN(T_R)|_{l_{p_j}[t_{j,m},1]} \cup D^2_{1,j}) \subset \gamma_1^\bullet$ which bounds that S^1 is obviously null in $\pi_2(M, \beta_0^\bullet)$, since it bounds $D^3 = DN(T_R)|_{l_{p_j}[t_{j,m},1]} \cup D^3_{1,j}, D^3_{1,j} \subset D^3 = DN(\gamma_1)|_{\gamma_1(v_j)}$ where $DN(T_R)$ denotes the normal disk bundle of T_R. Thus in the calculation of Θ, every meridian S^1 corresponds to $(0,0) \cdot [\alpha] = 0 \in \mathrm{Wh}_1(\pi_1 M; \mathbb{Z}_2 \times \pi_2 M)$, so they make no contributions. So $\Theta(f_\beta) = \sum_{i=1,\dots,k} (0, \beta_i^*) \cdot [\gamma_i^*]$, but for the same reason, $\beta_i^* = [D_i^B]^{\delta_i^B}$, that is, stretching the disk near the intersection point along l_{p_j} makes no difference in $\pi_2(M) = \pi_2(M, \beta_0^\bullet)$. Thus, $\Theta(f_\beta) = \sum_{i=1,\dots,k} (0, [D_i^B]^{\delta_i^B}) \cdot [\delta_i^B]$. □

Corollary 5.3. *For any $\sigma \in \pi_2 M$ with $w_2^M(\sigma) = 0$, $\forall \alpha \in \pi_1 M$, there is a half-unknotted immersed barbell $\beta = (R, S, \gamma)$ and $f_\beta \in \ker \Sigma \subset \pi_0 \mathcal{P}$ with its Cerf diagram being a single eye of (2,3)-handle pair resulting in the immersed barbell diffeomorphism w.r.t. β such that $\Theta(f_\beta) = (0, \sigma) \cdot [\alpha]$.*

Proof. For σ satisfying the condition, we first find R which is an immersed 2-sphere with $q \in R$, such that R represents $\sigma \in \pi_2(M, q)$. Since $w_2^M(\sigma) = 0$, we can induce some interior twists (see [9, section 1.3]) locally on R away from q to get R_σ which is a framed immersed 2-sphere representing the same $\sigma \in \pi_2(M, q)$. Then find a small embedded $\beta_0^\bullet : D^3 \hookrightarrow M$ away from R_σ with base point $*_0 \in S := \partial \beta_0^\bullet$. Fix $p \in int(\beta_0^\bullet)$. Find a path γ_1 from $*_0$ to p such that $int(\gamma_1) \cap (\beta_0^\bullet \sqcup R_\sigma) = \emptyset$, γ_1 represents $\alpha \in \pi_1(M, \beta_0^\bullet)$. Find γ_2 from p to q such that $int(\gamma_2) \cap (\beta_0^\bullet \sqcup R_\sigma) = \emptyset$, $(\gamma_1 * \gamma_2)^{-1}$ induces a natural isomorphism from $\pi_2(M, q)$ to $\pi_2(M, *_0)$: $\tau \to \tau^{(\gamma_1 * \gamma_2)^{-1}}$, we need $\sigma^{(\gamma_1 * \gamma_2)^{-1}} = \sigma$. Let $\gamma = \gamma_1 * \gamma_2$, then let $\beta = (R_\sigma, S, \gamma)$. By the above theorem, $\Theta(f_\beta) = (0, \sigma) \cdot [\alpha]$.

□

When there is an odd 2-sphere $\tau \in \pi_2 M$ (i.e. $w_2^M(\tau) = 1$), we can do more (the key point is that the π_2 component of Θ only depends on D_i, not on D_i'):

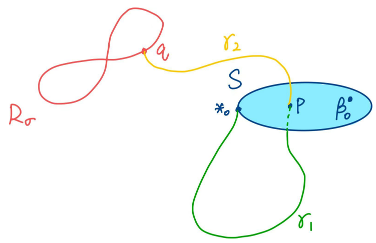

Figure 5.3 / An illustration of $\beta = (R_\sigma, S, \gamma = \gamma_1 * \gamma_2)$

Corollary 5.4. *If there exists $\tau \in \pi_2 M$ with $w_2^M(\tau) = 1$, then for any $\sigma \in \pi_2 M$, $\forall \alpha \in \pi_1 M$, there is a half-unknotted immersed barbell $\beta = (R, S, \gamma)$ and $f_\beta \in \ker \Sigma \subset \pi_0 \mathcal{P}$ with its Cerf diagram being a single eye of (2,3)-handle pair resulting in the immersed barbell diffeomorphism w.r.t. β such that $\Theta(f_\beta) = (0, \sigma) \cdot [\alpha]$.*

Proof. For $\sigma \in \pi_2 M$ with $w_2^M(\sigma) = 0$, the result just follows from Corollary 5.3. For $w_2^M(\sigma) = 1$. We find an immersed 2-sphere R_σ representing σ, and an immersed 2-sphere R_τ representing τ. Connect R_τ and R_σ with a tube along an embedded arc γ_0 disjoint from both R_σ and R_τ to get $R(\gamma_0)$. Then $R(\gamma_0)$ can be made into a framed immersed 2-sphere by inducing some interior twists locally. Find a small normal disk $\beta_0^\bullet$ on γ_0, let $*_0 \in S = \partial \beta_0^\bullet$ be the base point. Find another embedded path γ from $*_0$ to $p_\tau \in R_\tau$ away from $\beta_0^\bullet$ and $R(\gamma_0)$. See Figure 5.4 for an illustration. By suitably choosing γ_0 and γ, we can find $\beta = (R(\gamma_0), S, \gamma)$ such that $\Theta(f_\beta) = (0, \sigma) \cdot [\alpha]$.

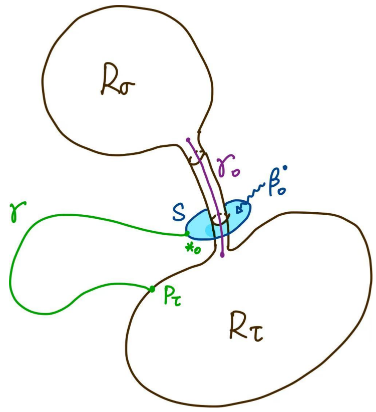

Figure 5.4 / An illustration for the proof of Corollary 5.4

□

Combining the above two corollaries we get:

Corollary 5.5. *For any $\sigma \in \pi_2 M$, $\forall \alpha \in \pi_1 M$, there is a half-unknotted immersed barbell $\beta = (R, S, \gamma)$ and $f_\beta \in \ker \Sigma \subset \pi_0 \mathcal{P}$ with its Cerf diagram being a single eye of (2,3)-handle pair resulting in the immersed barbell diffeomorphism w.r.t. β such that $\Theta(f_\beta) = (0, \sigma) \cdot [\alpha]$.*

Singh proved in [3] that for any compact M^4,

$$W = \langle (s, \sigma) \cdot [\gamma], s = 0 \text{ or } w_2^M(\sigma) \neq 0 \rangle \subset \mathrm{Wh}_1(\pi_1 M; \mathbb{Z}_2 \times \pi_2 M),$$

we have $W \subset \Theta(\ker \Sigma)$. In particular, when $M = X^3 \times I$, $W = \langle (0, \sigma) \cdot [\gamma] \rangle \subset \mathrm{Wh}_1(\pi_1 M; \mathbb{Z}_2 \times \pi_2 M)$ since all $\sigma \in \pi_2 M$ have $w_2^M(\sigma) = 0$. Then,

Corollary 5.6. *For $M = X^3 \times I$, all Hatcher-Wagoner invariants Θ realizable by Singh's procedure can be realized by compositions of f_β, where β is an immersed half-unknotted barbell.*

In particular, for $M = (X_1 \# X_2) \times I$ with X_i closed, orientable, aspherical 3-manifold, Singh used W and approximations for $K_3(\mathbb{Z}\pi_1 M)$ and $\Theta(\mathcal{J} \cap \ker \Sigma)$ to show that there's $K \subset \pi_0 \operatorname{Diff}_{PI}(M, \partial)$ and a surjection $K \to \bigoplus_{i \in \mathbb{N}} \mathbb{Z}$. Now by the above corollary, we have:

Corollary 5.7. *Let $M = (X_1 \# X_2) \times I$ with X_i closed, orientable, aspherical 3-manifold. In this case all $\sigma \in \pi_2 M = \mathbb{Z}[\pi_1 X_1 * \pi_1 X_2]$ can be realized by embedded S^2 with $w_2^M(\sigma) = 0$. Then*

$$\langle \textit{implanted half-unknotted barbell diffeomorphisms} \rangle \subset \pi_0 \operatorname{Diff}_{PI}(M, \partial)$$

is infinitely generated and of infinite $\mathbb{Z}$-rank.

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