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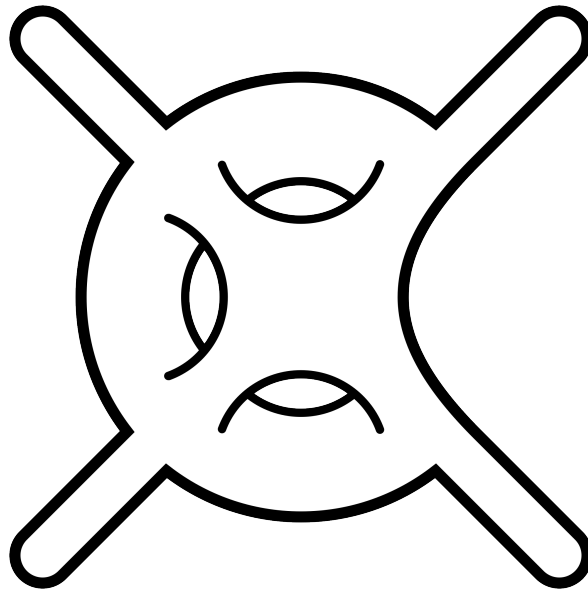
*Cerf Diagrams and
Hatcher-Wagoner Invariants
for Barbell Maps*

XXCC Unicorn Press

Cerf Diagrams and Hatcher-Wagoner Invariants for Barbell Maps

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*Dedicated to my beloved, Xiayu Tan,
for the beauty of her research
and the warmth of her heart.*

— *Yifei Fan*

Contents

Editor's Preface	ii
Introduction	iii
1 Cerf theory and parameterised handle constructions version	1
2 Twin twists and relation with barbell diffeomorphisms	4
3 Changing from (1,2)-handle pair to (2,3)-handle pair	8
4 Computation for the second Hatcher-Wagoner invariant	15
5 Generalizations to immersed barbell diffeomorphisms	18
Bibliography	23

Editor's Preface

It is with a heart full of excitement and pride that I present this work. I have had the privilege of witnessing Xiayu's remarkable transformation — from a budding topology enthusiast to a seasoned researcher capable of navigating the vast sea of literature to find truth.

Throughout this journey, I saw her moments of pure joy upon mastering a complex paper, as well as her frustrations when a seemingly brilliant construction turned out to be trivial. The day she told me, "I've done it," I felt a profound sense of happiness for her. She has truly come into her own as a researcher.

As a gift to celebrate her first-ever paper, I have prepared this work in the form of a journal article. It is my sincere wish that Xiayu will go on to publish in top-tier journals and continue to tackle even more challenging problems in the world of mathematics.

The Editor-in-Chief

I Introduction

Given a smooth oriented manifold X , let $\mathcal{P}(X)$ be the pseudo-isotopy group of X defined by

$$\mathcal{P}(X) := \{f \in \text{Diff}(X \times I) \mid f|_{\partial X \times I \cup X \times 0} = \text{id}\}$$

The pseudo-isotopy group is closely related to the diffeomorphism group of X by $\mathcal{P}(X) \xrightarrow{F} \text{Diff}(X, \partial) : f \rightarrow F_f = f|_{X \times 1}$, in that case we call that f results in F_f .

Hatcher and Wagoner first studied $\pi_0 \mathcal{P}(X)$ in their collected work [1] and it was later studied by Igusa in [2]. They found two invariants for the group, namely,

$$\Sigma : \pi_0(\mathcal{P}) \rightarrow \text{Wh}_2(\pi_1 X) \text{ and } \Theta : \ker \Sigma \rightarrow \text{Wh}_1(\pi_1 X; \mathbb{Z}_2 \times \pi_2 X) / \chi(K_3 \mathbb{Z}[\pi_1 X])$$

where we have $\text{Wh}_1(\pi_1 X; \mathbb{Z}_2 \times \pi_2 X) = (\mathbb{Z}_2 \times \pi_2 M)[\pi_1 M] / (\beta \cdot [1], \alpha \cdot [\sigma] - \alpha^\tau \cdot [\tau \sigma \tau^{-1}], \alpha, \beta \in \mathbb{Z}_2 \times \pi_2 M, \tau, \sigma \in \pi_1 M)$. These two maps are always well-defined whenever $\dim X \geq 4$, and they showed that Σ is always surjective when $\dim X \geq 5$, and Θ is surjective when $\dim X \geq 5$ and bijective when $\dim X \geq 6$. In [3] Singh studied the partial images of Σ and Θ in dimension 4 and showed that both Σ and Θ are stably surjective.

The pseudo-isotopy group is closely related to the diffeomorphism group, for there is a natural fiber bundle

$$\mathcal{J} = \text{Diff}(X \times I, \partial) \rightarrow \mathcal{P}(X) \xrightarrow{F} \text{Diff}_{PI}(X, \partial)$$

where $\text{Diff}_{PI}(X, \partial) \subset \text{Diff}(X, \partial)$ denotes all the diffeomorphisms of X which are pseudo-isotopic to identity. Thus we have *induced* Hatcher-Wagoner invariants

$$\Sigma : \pi_0 \text{Diff}_{PI}(X, \partial) \rightarrow \text{Wh}_2(\pi_1 X) / \Sigma(\mathcal{J})$$

$$\Theta : \pi_0 F(\ker \Sigma) \rightarrow \text{Wh}_1(\pi_1 X; \mathbb{Z}_2 \times \pi_2 X) / (\Theta(\mathcal{J} \cap \ker \Sigma) + \chi(K_3 \mathbb{Z}[\pi_1 X]))$$

on the diffeomorphism group of X , where $\pi_0 F(\ker \Sigma) \subset \pi_0 \text{Diff}_{PI}(X, \partial)$

Recently, Gabai, Budney, Gay and Hartman constructed the 4-dimensional barbell diffeomorphism (a specific nontrivial element in $\pi_0 \text{Diff}(\mathcal{B}_{2,2}^4, \partial)$ following the notations in [4]) and studied the implanted barbell diffeomorphisms in a general X^4 , especially in $S^1 \times D^3$ ([5]) and S^4 ([6]). One of the remarkable results is that Budney and Gabai constructed an infinitely generated subset in $\pi_0(\text{Diff}(S^1 \times D^3, \partial) / \text{Diff}(D^4, \partial))$, more explicitly, an inclusion

$$\bigoplus_{k \geq 4 \in \mathbb{N}} \mathbb{Z} \rightarrow \pi_0(\text{Diff}(S^1 \times D^3, \partial) / \text{Diff}(D^4, \partial)) : 1_k \rightarrow \delta_k$$

where $\{\delta_k\}_{k \in \mathbb{N}}$ are specific implanted barbell diffeomorphisms in $S^1 \times D^3$ which we will carefully

describe later in this paper.

Suppose we have an implanted barbell in a general X^4 , i.e. $\beta : S^2 \times D^2 \natural S^2 \times D^2 \hookrightarrow X$, and therefore an implanted barbell diffeomorphism in $\text{Diff}(X, \partial)$, it is known that (see [4, Proposition 2.6]) if one of the implanted core spheres is unknotted in X^4 , then the implanted barbell diffeomorphism is pseudo-isotopic to identity. One natural question is:

Question. How to compute the induced Hatcher-Wagoner invariants for the given half-unknotted implanted barbell diffeomorphism?

Once we get an $f_\beta \in \pi_0\mathcal{P}$ resulting in that implanted barbell diffeomorphism, $\Sigma(f_\beta), \Theta(f_\beta)$ are representatives of the two induced Hatcher-Wagoner invariants. In this article, we'll construct two pseudo-isotopies $g_\beta, f_\beta \in \pi_0\mathcal{P}$, both resulting in the implanted barbell diffeomorphism with respect to β . g_β has a Cerf diagram containing a single eye of (1,2)-handle pair and f_β has a Cerf diagram containing a single eye of (2,3)-handle pair. The main result will give an explicit formula for the Hatcher-Wagoner invariants of f_β , which is stated as follows:

Theorem 0.1. *For a half-unknotted implanted barbell $\beta = (R_0, S, \gamma)$ with $S = \partial\beta_0^\bullet$ where $\beta_0^\bullet : D^3 \hookrightarrow M$, by finger-pushing R_0 along the arc, we can make γ short enough such that $\text{int}(\beta_0^\bullet) \cap \gamma = \emptyset$. Now suppose that $\beta_0^\bullet \cap R_0 = \bigsqcup_{i=1}^k S_i^1$. Choose $p_i \in S_i^1$ and let $*_0 = \gamma \cap S$ be the base point. For each i , find a path $\delta_i^B \subset \beta = (R_0, S, \gamma)$ which is a path from $*_0$ to $p_i \in S_i^1$. Also, for each i , S_i^1 divides R_0 into two embedded disks D_i and D'_i , where D'_i is the one connected to the arc γ . Let $D_i^B = D_i$ (see Figure 1 for an illustration). Then there is a pseudo-isotopy $f_\beta \in \pi_0\mathcal{P}$ resulting in that implanted barbell diffeomorphism with the Cerf diagram of f_β being a single eye of (2,3)-handle pair such that:*

$$\Sigma(f_\beta) = 0, \Theta(f_\beta) = \sum_{i=1, \dots, k} (0, [D_i^B]^{\delta_i^B}) \cdot [\delta_i^B]$$

Here we identify $\pi_i(M, *_0)$ with $\pi_i(M, \beta_0^\bullet)$ so that $[D_i] \in \pi_2(M, \beta_0^\bullet)$ and $\delta_i^B \in \pi_1(M, \beta_0^\bullet)$.

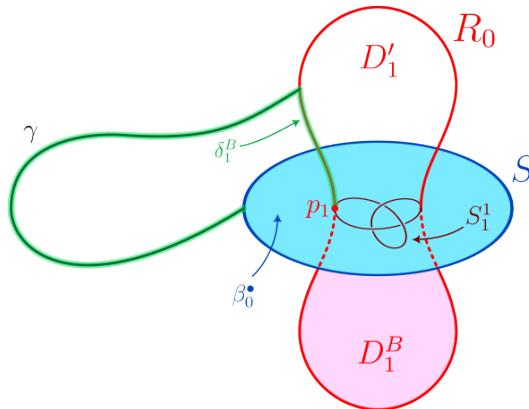


Figure 1 / A general barbell and data needed to compute Hatcher-Wagoner invariants

Remark 0.2. In particular, when the given barbell $\beta = (R_0, S, \gamma)$ with $S = \partial\beta_0^\bullet$ and $\gamma : [0, 1] \rightarrow M, \gamma(0) = *_0 = \gamma \cap S$ satisfies $R_0 \cap \beta_0^\bullet = \emptyset$, suppose $\gamma|_{(0,1)} \cap \beta_0^\bullet = \{p_i\}_{i=1}^k$ with $p_i = \gamma(t_i)$, then the $f_\beta \in \pi_0\mathcal{P}$ we constructed in the theorem satisfies:

$$\Sigma(f_\beta) = 0, \Theta(f_\beta) = (0, [R_0^\gamma]) \sum_{i=1}^k [\gamma_i]$$

where $\gamma_i := \gamma|_{[0,t_i]} \in \pi_1(M, \beta_0^\bullet)$ and $[R_0^\gamma] \in \pi_2(M, \beta_0^\bullet)$ is obtained by pulling R_0 back to $*_0$ along γ . ┘

Remark 0.3. Along the way we also compute the Hatcher-Wagoner invariants for g_β : $\Sigma(g_\beta) = 0, \Theta(g_\beta) = -\bar{\Theta}(f_\beta)$, where $\bar{\cdot}$ is an involution on $\text{Wh}_1(\pi_1, \mathbb{Z}_2 \times \pi_2)$ which is described in both [1] and [3, section 9.1]. We review the involution here:

$$\bar{\cdot} : (n, \sigma) \cdot [\gamma] \rightarrow (n + w_2^X(\sigma), -w_1^X(\gamma)\sigma^{\gamma^{-1}}) \cdot [\gamma^{-1}],$$

where $w_2^X(\sigma) \in \{0, 1\}$ is the second Stiefel-Whitney class of the normal bundle of σ in X , and $w_1^X(\gamma) \in \{-1, 1\}$ is the first Stiefel-Whitney class of the normal bundle of γ in X . ┘

Remark 0.4. In particular, all $\{f_{\delta_k}\}_{k \in \mathbb{N}}$ in $\mathcal{P}(S^1 \times D^3)$ lie in the kernel of Σ and Θ , this answers a question raised by Powell in [7, Question 12.6]. But $\{\delta_k, k \geq 4\}$ are nontrivial diffeomorphisms of $S^1 \times D^3$, so $[f_{\delta_k}, k \geq 4] \neq 0 \in \pi_0\mathcal{P}(S^1 \times D^3)$, which equivalently means that $\{f_{\delta_k}, k \geq 4\}$ are nontrivial pseudo-isotopies of $S^1 \times D^3$ which can not be detected by Hatcher-Wagoner invariants. ┘

We generalize the calculation to *half-unknotted immersed barbell diffeomorphisms* which will be defined in Chapter 5. The result is completely the same:

Theorem 0.5. *For a half-unknotted immersed barbell $\beta = (R_0, S, \gamma)$ with $\beta_0^\bullet : D^3 \hookrightarrow M, S = \partial\beta_0^\bullet$, perturb self-intersections of R_0 away from β_0^\bullet . By finger-pushing R_0 along arc, we can make γ short enough such that $\text{int}(\beta_0^\bullet) \cap \gamma = \emptyset$. Now suppose that $\beta_0^\bullet \cap R_0 = \bigsqcup_{i=1}^k S_i^1$. Choose $p_i \in S_i^1$ and let $*_0 = \gamma \cap S$ be the base point. For each i , find a path $\delta_i^B \subset \beta = (R_0, S, \gamma)$ which is a path from $*_0$ to $p_i \in S_i^1$. Also, for each i , S_i^1 divides R_0 into two embedded disks D_i and D'_i , where D'_i is the one connected to the arc γ . Let $D_i^B = D_i$. Then the $f_\beta \in \pi_0\mathcal{P}$ we constructed which results in the immersed barbell diffeomorphism w.r.t. β satisfies:*

$$\Theta(f_\beta) = \sum_{i=1, \dots, k} (0, [D_i^B]^{\delta_i^B}) \cdot [\delta_i^B]$$

Here we identify $\pi_i(M, *_0)$ with $\pi_i(M, \beta_0^\bullet)$ so that $[D_i] \in \pi_2(M, \beta_0^\bullet)$ and $\delta_i^B \in \pi_1(M, \beta_0^\bullet)$.

From the main theorem we deduce some corollaries:

Corollary 0.6. *For any $\sigma \in \pi_2 M, \forall \alpha \in \pi_1 M$, there is a half-unknotted immersed barbell $\beta = (R, S, \gamma)$ and $f_\beta \in \ker \Sigma \subset \pi_0\mathcal{P}$ with its Cerf diagram being a single eye of (2,3)-handle pair resulting in the immersed barbell diffeomorphism w.r.t. β such that $\Theta(f_\beta) = (0, \sigma) \cdot [\alpha]$.*

In particular, using results deduced by Singh in [3] where Singh showed that for $M = (X_1 \# X_2) \times I$ with X_i a closed, oriented, aspherical 3-manifold, there's $K \subset \pi_0 \text{Diff}_{PI}(M, \partial)$ and a surjection $K \rightarrow \bigoplus_{i \in \mathbb{N}} \mathbb{Z}$, we further show that we can let

$$K = \langle \text{implanted half-unknotted barbell diffeomorphisms} \rangle,$$

that is:

Corollary 0.7. *Let $M = (X_1 \# X_2) \times I$ with X_i closed, orientable, aspherical 3-manifold. In this case all $\sigma \in \pi_2 M = \mathbb{Z}[\pi_1 X_1 * \pi_1 X_2]$ can be realized by embedded S^2 with $w_2^M(\sigma) = 0$. Then*

$$\langle \text{implanted half-unknotted barbell diffeomorphisms} \rangle \subset \pi_0 \text{Diff}_{PI}(M, \partial)$$

is infinitely generated and of infinite \mathbb{Z} -rank.

The paper is organized as follows: In Chapter 1 we recall the Cerf theory and parametrised handle constructions version of it, especially we explain how a loop of handle constructions of $X \times I$ can result in a pseudo-isotopy $f \in \pi_0 \mathcal{P}$ of X , and how to see the resulting $F_f \in \text{Diff}(X, \partial)$ directly from the loop of handle constructions. In Chapter 2 we recall the general notion of Montesino twins in X^4 and the resulting twin twists in $\text{Diff}_0(X, \partial)$ which were studied for $X = S^4$ in [6]. Then we show that if one of the spheres in the Montesino twins is unknotted, then $\exists f \in \mathcal{P}(X)$ resulting in that twin twist with Cerf diagram being a loop of a single cancelling (1,2)-handle pair. Then we show that any implanted barbell diffeomorphism is a twin twist, so together with the above result, for a specific implanted barbell β , we find $g_\beta \in \mathcal{P}(X)$ with Cerf diagram being a loop of a single cancelling (1,2)-handle pair resulting in that implanted barbell diffeomorphism. In Chapter 3 we follow the essential lemma [6, Lemma 17] to develop a technique of *one parameter version of 0-framed and dotted replacement* to change the pseudo-isotopy $g_\beta \in \pi_0 \mathcal{P}$ with Cerf diagram being the above loop of cancelling (1,2)-handle pair to $f_\beta \in \pi_0 \mathcal{P}$ with Cerf diagram being a loop of cancelling (2,3)-handle pair. Therefore we get a loop of cancelling (2,3)-handle pair resulting in implanted barbell diffeomorphism of β . In Chapter 4 we use the technology in Chapter 3 to complete the computations of second Hatcher-Wagoner invariant for a half-unknotted implanted barbell diffeomorphism. In Chapter 5 we generalize to *immersed barbell diffeomorphisms* and compute the induced Hatcher-Wagoner invariants for a half-unknotted immersed barbell diffeomorphism. As corollaries, we deduce some realization results on what induced Hatcher-Wagoner invariants can be realized by immersed half-unknotted barbell diffeomorphisms.

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The work was carried out during the author's fourth year in Qiuzhen College, Tsinghua University, and thus she expresses her deep gratitude to the institution.

1 Cerf theory and parameterised handle constructions version

We have defined the pseudo-isotopy group $\mathcal{P}(X)$ and want to study $\pi_0\mathcal{P}(X)$. Cerf discovered a great correspondence:

$$\pi_0\mathcal{P} \cong \pi_1(\mathcal{F}, \mathcal{E})$$

where $\mathcal{F} = \mathcal{F}(X) := \{f : X \times I \rightarrow I \mid f = \text{standard projection on a neighborhood of } \partial(X \times I)\}$ and $\mathcal{E} = \mathcal{E}(X) \subset \mathcal{F}(X)$ contains all such f that have no critical points at all. The explicit correspondence is: Given $f \in \mathcal{P}(X)$, let p be the standard projection from $X \times I$ to I . Then $p \circ f \in \mathcal{E}(X)$ since f is a diffeomorphism of $X \times I$. It is not hard to show that $\mathcal{F}(X)$ is contractible, therefore we can choose a path $\gamma = \{\gamma_t : t \in I\}$ in \mathcal{F} with $\gamma_0 = p, \gamma_1 = p \circ f$ both in \mathcal{E} . Then γ is the corresponding element in $\pi_1(\mathcal{F}, \mathcal{E})$. Conversely, given $f_t : X \times I \rightarrow I$ starting at $f_0 = p$ and ending at $f_1 = q \in \mathcal{E}$, consider the gradient vector field V_q of q on $X \times I$, integrating V_q gives a diffeomorphism on $X \times I : (x, t) \rightarrow \phi_{V_q}(x, 0)(t)$.

Then we reduce the question to the study of *one parameter family of functions* on $X \times I$, namely, $f_t : X \times I \rightarrow I$ with f_0, f_1 no critical points at all. Like what we did in Morse theory, we can perturb the path fixing f_0, f_1 such that for all but finitely many $t \in I$, f_t is a Morse function with ordered, distinct critical values. At the exceptional points, there appears either an $(i, i + 1)$ -birth/death point (two critical points of index i and $i + 1$ have the same critical value, but by the ordering condition, they must cancel in one direction) or an (i, i) -crossing point (two non-degenerate critical points of same index i with crossing critical values). The Cerf diagram records the changes of critical values in $f_t : X \times I$. An example is shown in Figure 1.1.

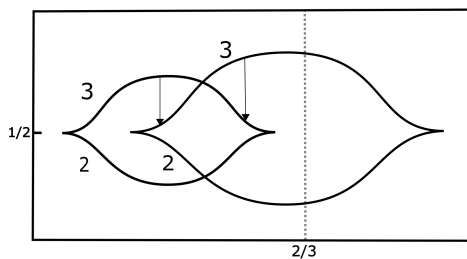


Figure 1.1 / This is a Cerf diagram with 2 eyes, each eye begin from the birth to the end. All births and ends happen in $g_t^{-1}(1/2)$. Except the four birth/death points, every g_t is Morse. For example, $g_{2/3}$ has two non-degenerate critical points of index 2 and 3. There are two crossing points in the diagram. Every arrow denotes a handle slide between critical points with the same index.

In this paper we are trying to find a Cerf diagram for f_β with only one eye of index 2 and 3. Then let $g_t, t \in I$ be the corresponding one parameter family of functions on $X \times I$ with $g_0 = p, g_1 = p \circ f_\beta$. If we successfully find such a family g_t with such a Cerf diagram, then the only things happen here are as follows: For some small ϵ , g_ϵ births a degenerate critical point (standard local model is $g_\epsilon = x^3 - y_1^2 - y_2^2 + y_3^2 + y_4^2$ near the origin $p = (0, 0, 0, 0, 0)$ with $g_\epsilon(p) = 1/2$), and for $t > \epsilon$, the critical point splits into two non-degenerate critical points of

indices 2 and 3 (standard local model is $g_{\epsilon+s} = x^3 - sx - y_1^2 - y_2^2 + y_3^2 + y_4^2$). Right after the birth point at $g_\epsilon^{-1}(1/2)$, the level set is $g_{\epsilon+s}^{-1}(1/2) = X \# S^2 \times S^2$, with the dual sphere of the unique 2-handle being $* \times S^2$ and the attaching sphere of the unique 3-handle being $S^2 \times *$. Then the two spheres isotope in $X \# S^2 \times S^2$ during the interval $t \in (\epsilon, 1 - \epsilon)$. They may intersect each other at more than one point during the isotopy, but finally, right before $t = 1 - \epsilon$, they return to the dual position, i.e. intersect at exactly one point. Then general Morse theory tells that the (2,3)-handle pair can be cancelled, so a death occurs at $t = 1 - \epsilon$. But via an isotopy, we can assume one of the spheres is fixed throughout. In short, the $\{g_t, t \in I\} \in \pi_1(\mathcal{F}, \mathcal{E})$ is completely governed by an element in $\pi_1(\text{Emb}(S^2 \times D^2, X \# S^2 \times S^2), \text{Emb}_0(S^2 \times D^2, X \# S^2 \times S^2))$ where $\text{Emb}_0(S^2 \times D^2, X \# S^2 \times S^2)$ denotes all the embedded $S^2 \times D^2$ such that $S^2 \times 0$ transversely intersect $* \times S^2$ at a single point.

A similar story holds for a Cerf diagram with only one eye of index 1 and 2. To be explicit, $\{g_t, t \in I\} \in \pi_1(\mathcal{F}, \mathcal{E})$ is completely governed by an element in $\pi_1(\text{Emb}(S^1 \times D^3, X \# S^1 \times S^3), \text{Emb}_0(S^1 \times D^3, X \# S^1 \times S^3))$ with similar notation for $\text{Emb}_0(S^1 \times D^3, X \# S^1 \times S^3)$.

In particular, whenever we have a loop of framed embedded S^i in $X \# S^i \times S^{n-i}$, which means the element lies not only in the relative π_1 , but also in $\pi_1(\text{Emb}(S^i \times D^{n-i}, X \# S^i \times S^{n-i}), *)$, tracing the flow lines it is not hard to show that the resulting diffeomorphism on X is just the following composition:

$$\pi_1(\text{Emb}(S^i \times D^{n-i}, X \# S^i \times S^{n-i}), *) \rightarrow \pi_0 \text{Diff}(X \setminus \nu S^{n-i-1}, \partial) \xrightarrow{\text{Uid}_{\nu S^{n-i-1}}} \pi_0 \text{Diff}(X, \partial)$$

where the first map is the isotopy extension at $t = 1$ and removing $\nu S^i = \nu(S^i \times *)$ in $X \# S^i \times S^{n-i}$, which is diffeomorphic to $X \setminus \nu S^{n-i-1}$, where the S^{n-i-1} denotes the dual sphere of the $(i+1)$ -handle.

Just like “Building a manifold X from a Morse function is equivalent to finding a handle decomposition for X ”, we have an equivalent version for pseudo-isotopy: *Finding a one-parameter family of functions on $X \times I$ is equivalent to finding a one-parameter family of handle decompositions for $X \times I$.*

Thus if we have a loop $\mathcal{H} = \{H_t, t \in I | H_0 = H_1\}$ of handle decompositions of $X \times I$ which starts and ends at a standard cancelling position which can be made null, by the above statement, this corresponds to an element in $\pi_1(\mathcal{F}, \mathcal{E}) = \pi_0 \mathcal{P}$, the resulting manifold is Z_t which is diffeomorphic to $X \times I$. Since $H_0 = H_1$, we have a natural way to identify Z_0 with Z_1 (since they are totally the same manifold). Then we build a cobordism from $S^1 \times X$ to $Y = X \times I / ((x, 0) \sim (f_{\mathcal{H}}(x), 1))$ where $f_{\mathcal{H}}$ is the resulting diffeomorphism on X with respect to $\mathcal{H} \in \pi_1(\mathcal{F}, \mathcal{E})$. In handlebody language, we describe what $f_{\mathcal{H}}$ is: Each H_t gives a series of surgeries $X = X_{0,t} \rightarrow X_{1,t} \rightarrow \dots \rightarrow X_{n,t}$, $X_{n,t}$ is diffeomorphic to X but not in a natural way. But again, since $H_0 = H_1$, we have $X_{n,0} = X_{n,1} \cong X$. But as t varies, since H_t moves smoothly, every attaching region moves smoothly, thus we have a diffeomorphism $\phi_{m,t} : X_{m,0} \rightarrow X_{m,t}, \forall m, t$. In particular $\phi_{n,1} = f_{\mathcal{H}} : X = X_{n,0} \rightarrow X_{n,1} = X$ is the desired diffeomorphism on X .

In particular, when \mathcal{H} is a loop of a single $(i, i + 1)$ -cancelling pair, that is, $Z_t = X \times I \cup h_{i,t} \cup h_{i+1,t}$. Since $h_{i,0}$ and $h_{i+1,0}$ form a standard cancelling pair, the attaching sphere of $h_{i,0}$ must be unknotted. Then let $X_{1,0} = (X \setminus \nu S^{i-1}) \cup D^i \times S^{n-i} = X \# S^i \times S^{n-i}$ be the manifold after doing surgery on h_i . Up to an isotopy, we can assume $h_{i,t} = h_{i,0}$ throughout, so $X_{1,t} = X_{1,0} = X \# S^i \times S^{n-i}$. The only t -dependent data is the attaching region of $h_{i+1,t}$, which corresponds to $\pi_1(\text{Emb}(S^i \times D^{n-i}, X_{1,0}), *)$, then one can directly see that the resulting $f_{\mathcal{H}} : X = X_{2,0} \rightarrow X_{2,1} = X$ is the composition map mentioned above.

2 Twin twists and relation with barbell diffeomorphisms

First we derive a standard model for Montesino twins: Consider $X = S^1 \times S^1 \times D^2$ with boundary $\partial X = S^1_l \times S^1_S \times S^1_R$, where $S^1_l = S^1 \times 1 \times 1$, $S^1_S = -1 \times S^1 \times 1$, $S^1_R = -1 \times 1 \times \partial D^2$. Then we do surgery on $\nu(1 \times S^1 \times 0)$, and let $X' = (X \setminus \nu(1 \times S^1 \times 0)) \cup D^2 \times S^2$ be the resulting manifold. Then the core $T_R = (S^1 \times S^1 \times 0)$ of X becomes $R = (T_R \setminus I \times S^1) \cup \partial I \times D^2$, which is a 2-sphere with normal sphere S^1_R , and $S = 0 \times S^2 \subset D^2 \times S^2$ is another 2-sphere with normal sphere S^1_S . Moreover, R intersects with S transversely at 2 points. The resulting manifold X' is just a tubular neighborhood of $R \cup S$ with the same boundary $\partial X' = \partial X = S^1_l \times S^1_S \times S^1_R$. Centered at $* = (-1, 1, 1)$, S^1_l is homotopic to a longitude of X' , that is, $\pi_1(\partial X', *) \rightarrow \pi_1(X', *) \cong \pi_1(S^2 \vee S^2 \vee S^1, *) = \mathbb{Z}$ sends $[S^1_l]$ to 1 after a suitable orientation. An easy way to see that is, $\pi_1(\partial X, *) \rightarrow \pi_1(X, *)$ simply kills S^1_R and sends others via identity, and surgering X to X' simply kills S^1_S since it's a parallel copy of $1 \times S^1 \times 0$, that means $\pi_1(X') = \pi_1(X)/[S^1_S] = \mathbb{Z}[S^1_l]$. Moreover, a simple way to construct a longitude in X' is to consider $\{p, q\} = R \cap S$, and choose $\gamma_R \subset R, \gamma_S \subset S$ connecting p and q , then the resulting $\gamma_R \cup_{p,q} \gamma_S$ is a longitude of X' .

A standard twin twist is a Dehn twist near the boundary of X' , that is, choose a neighborhood of $\partial X' = I \times S^1_l \times S^1_S \times S^1_R$, and let $\phi_D \in \text{Diff}(I \times S^1_l, \partial)$ be the standard Dehn twist, then twin twist τ is identity-extension of $\phi_D \times \text{id} \times \text{id} \in \text{Diff}(\nu(\partial X'), \partial)$ in $\text{Diff}(X', \partial)$.

We provide a more natural way to construct the twin twist τ : Consider a loop of S^1 in X : $\gamma_t : S^1 \rightarrow e^{2\pi it} \times S^1 \times 0$, the isotopy extension of γ_t at $t = 1$ induces a diffeomorphism on X which fixes γ_0 . Moreover, it can be made to fix the whole core T_R , that is, it induces a diffeomorphism on $X \setminus \nu T_R = \nu(\partial X)$, which is exactly $\phi_D \times \text{id} \times \text{id}$. Therefore τ is the induced diffeomorphism after surgering γ_0 .

Definition 2.1. A *Montesino twin* in M^4 is a pair $W = (R, S)$ where R, S are two embedded S^2 in M with trivial normal bundle and R intersects S transversely at 2 points. Equivalently, a Montesino twin in M^4 is an embedding $i_W : X' \hookrightarrow M$. The twin twist induced by W is just the implanted diffeomorphism τ , denoted by τ_W .

Any twin twist is induced by a *parameterised surgery of index one* (see [8] for details) we will briefly describe below:

Definition 2.2. Given a framed embedded S^2 in M^4 , that is, $\nu S : S^2 \times D^2 \rightarrow M^4$, let $M_{\nu S} := (M \setminus \nu S) \cup D^3 \times S^1$ be the manifold obtained by performing surgery on S . We say a diffeomorphism ϕ of M is induced by a *parameterised surgery of index one* if ϕ is in the image of $\text{ps}_{\nu S}$:

$$\text{ps}_{\nu S} : \pi_1(\text{Emb}(\nu S^1, M_{\nu S}), i_0) \rightarrow \pi_0 \text{Diff}(M \setminus \nu S, \partial) \xrightarrow{\cup \text{id}_{\nu S}} \pi_0 \text{Diff}(M, \partial)$$

where $i_0 = 0 \times S^1 \subset M_{\nu S}$ with natural framing is the base point of the embedding space and the first map is induced by isotopy extension.

Proposition 2.3. *For any Montesino twin $W = (R, S)$ in M , the twin twist τ_W is induced by a parameterised surgery of index one on S .*

Proof. By the standard model of Montesino twins we have constructed, we do surgery on S we get an embedding $i : X = S^1 \times S^1 \times D^2 = \nu T^2 \rightarrow M_{\nu S}$ with $i_0 = i(1 \times S^1 \times 0)$. By the second way we describe twin twist τ , $\tau_W = \text{ps}_{\nu S}(\gamma_t)$ where $\gamma_t : S^1 \rightarrow i(e^{2\pi it} \times S^1 \times 0) \subset M_{\nu S}$. \square

Proposition 2.4. *When $W = (R, S)$ with S unknotted, then there exists a loop $\mathcal{H} \in \pi_0 \mathcal{P}$ of handle decompositions of $M \times I$ with a single $(1,2)$ -cancelling pair resulting in τ_W . In particular, τ_W is pseudo-isotopic to identity with the corresponding Cerf diagram being a single eye of $(1,2)$ -handle pair.*

Proof. When S is an unknotted S^2 , $M_{\nu S} = M \# S^3 \times S^1 = M \# S^1 \times S^3$, this can be regarded as attaching a trivial one handle. Then Prop 3.3 proved that τ_W is induced by an element $\gamma \in \pi_1(\text{Emb}(\nu S^1, M \# S^1 \times S^3), i_0)$ where $i_0 = S^1 \times * \in S^1 \times S^3$. Then $\tau_W = \phi_\gamma(1)|_{M \# S^1 \times S^3 \setminus \nu S^1} \cup \text{id}_{\nu S^2} = f_{\mathcal{H}}$ where $\phi_\gamma(t)$ denotes the time t isotopy extension on $M \# S^1 \times S^3$. \square

Then we recall the definition of an (implanted) barbell diffeomorphism and prove that any implanted barbell diffeomorphism is a special twin twist.

Definition 2.5. The *standard barbell diffeomorphism* is defined as follows: Consider D^4 and 2 disjoint properly embedded arcs I_1, I_2 in it. Let I_1 rotate around a normal sphere $S^2 \times * \subset SN(I_2) = S^2 \times I_2$ and comes back. This isotopy extension induces an element in $\text{Diff}(D^4 \setminus (\nu I_1 \cup \nu I_2) = S^2 \times D^2 \natural S^2 \times D^2, \partial)$. The *implanted barbell diffeomorphism* is obtained by implanting a barbell $\beta = (S_1, S_2, \gamma)$, i.e. S_1, S_2 two framed embedded S^2 with an arc connecting them, and extending the barbell diffeomorphism by id on $M \setminus \nu \beta$.

Proposition 2.6. *Let $\beta = (R_0, S, \gamma)$ be an implanted barbell in M . Finger-push R_0 along γ into S to get another sphere R , then R intersects S transversely at 2 points. Then the implanted barbell diffeomorphism by β is isotopic to twin twist τ_W where $W = (R, S_0)$.*

Proof. We first describe τ_W with $W = (R, S)$: After surgering along S_0 , we get an embedding $\nu T_R \rightarrow M_{\nu S}$ where T_R is a torus which has a framing induced by νR_0 , this corresponds to an element $[\nu T_R]$ in $\pi_1(\text{Emb}(\nu S^1, M_{\nu S}), *)$, then we have:

$$\begin{array}{ccc} \pi_1(\text{Emb}(\nu S^1, M_{\nu S}), *) & \longrightarrow & \pi_0 \text{Diff}(M_{\nu S} \setminus \nu S^1 = M \setminus \nu S, \partial) \xrightarrow{\cup \text{id}_{\nu S}} \pi_0 \text{Diff}(M, \partial) \\ \uparrow & & \uparrow \\ \pi_1(\text{Emb}(\nu I_1, D^2 \setminus \nu I_2 = S^2 \times D^2), *) & \longrightarrow & \pi_0 \text{Diff}(S^2 \times D^2 \natural S^2 \times D^2, \partial) \end{array}$$

The first line is just $ps_{\nu S}$ so by Prop 3.3 it sends $[\nu T_R]$ to τ_W . On the other hand, when S^1 goes around T_R , it just goes along γ , winds R around and then comes back. This is exactly what I_1 does in the standard barbell. Thus we just implant $(I_1, S^2 \times D^2)$ into $M_{\nu S}$ (see Figures in Example 2.8 below). The resulting diffeomorphism in $M_{\nu S} \setminus \nu S^1 = M \setminus \nu S$ is the barbell diffeomorphism with implanted $\beta = (R_0, \text{normal sphere of } S^1 = S, \gamma)$, when extending to M it's still the same barbell diffeomorphism. \square

Combining the above two propositions we get:

Corollary 2.7. *For any implanted $\beta = (R, S, \gamma)$ with S unknotted in M , the implanted barbell diffeomorphism is pseudo-isotopic to identity by $g_\beta \in \pi_0 \mathcal{P}$ with the corresponding Cerf diagram being a single eye of $(1,2)$ -handle pair. Moreover, it's induced from an element $[\nu T_R] \in \pi_1(\text{Emb}(S^1 \times D^3, M \# S^1 \times S^3), *)$.*

Example 2.8. Here we draw the corresponding loop in $\pi_1(\text{Emb}(S^1, S^1 \times D^3 \# S^1 \times S^3), *)$ (for simplicity I omit the framing but it's just the natural framing induced by $\nu T_R|_{S^1}$ and $\nu(S^1, T_R)$) which corresponds to Gabai's constructions δ_k (see Figures 2.1 to 2.3).

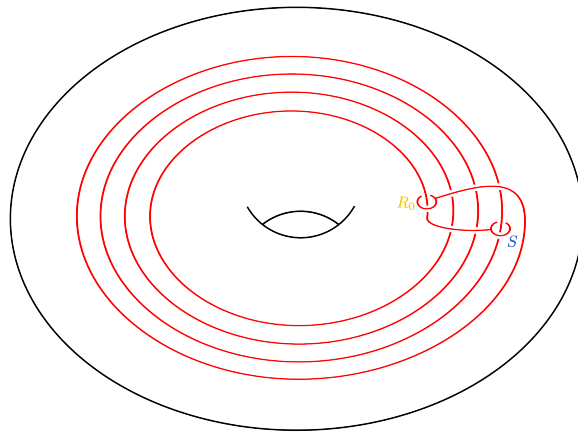


Figure 2.1 / The implanted barbell δ_4 in $M = S^1 \times D^3$

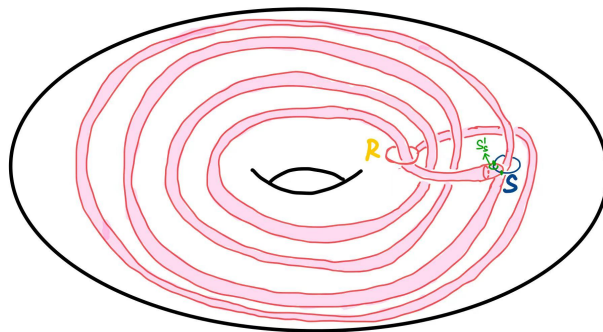


Figure 2.2 / Finger-push R_0 to get R such that $R \cap S = 2$ points

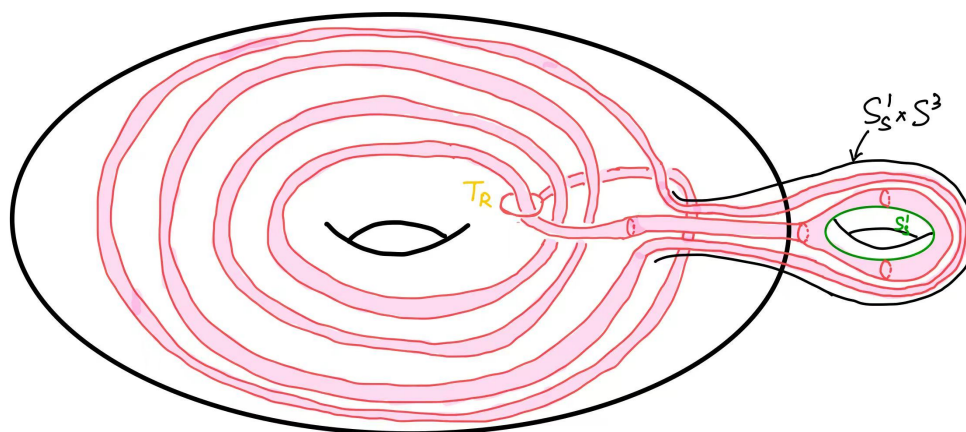


Figure 2.3 / Do surgery along S to get an embedded torus T_R in $S^1 \times D^3 \# S^1 \times S^3$ with the standard S^1 (green one), this T_R determines an element in $\pi_1(\text{Emb}(S^1, S^1 \times D^3 \# S^1 \times S^3), *)$ which is a loop of handle decompositions $\mathcal{H} = \{H_t, t \in S^1\}$ (a loop of (1,2)-handle pair) of $S^1 \times D^3 \times I$ which results in the barbell diffeomorphism $\delta_4 = \tau_W$ where $W = (R, S)$.

3 Changing from (1,2)-handle pair to (2,3)-handle pair

The author first heard the excellent idea from [6]. This is just the 5-dimensional Kirby calculus, where in dimension 4, we use dotted unknotted $S^1 = \partial D^2 \subset M^3$ to represent a 1-handle, and framed $S^1 \subset M$ to represent 2-handle. The central idea is that when the framed S^1 is unknotted, *swapping the dotted circle and the 0-framed circle* changes the cobordism but results in the same boundary manifold.

In this chapter we first describe the dotted version of trivial 1 handles and trivial 2 handles in a 5-dimensional handle construction H on $X^5 = M \times I$, then derive the strategy of *dotted and 0-framed replacement*. After that, we move on to the one-parameter versions of both, namely, dotted version of a Cerf diagram containing a single eye of (1,2)-handle pair which corresponds to an element in $\pi_1(\text{Emb}(S^1 \times D^3, M \# S^1 \times S^3), \text{Emb}_0(S^1 \times D^3, M \# S^1 \times S^3))$, and carefully develop the *one-parameter version of dotted and 0-framed replacement*.

Consider the trivial cobordism $Z = M \times I$ from M to M , if we attach a one handle on top of Z , we get $Z_1 = Z \cup h_1$ which is a cobordism from M to $M_1 = (M \setminus S^0 \times D^4) \cup D^1 \times S^3 = M \# S^1 \times S^3$. If we push the one handle into original $Z = M \times I$, we know that it's the same as cutting a neighborhood of an unknotted properly embedded D^3 in $M \times I$, that is, $Z_1 = M \times I \setminus \nu D_{in}^3$ where D_{in}^3 is obtained as follows: Choose an embedding $i : D^3 \rightarrow M \times I$, fix $i(\partial D^3)$ then push $i(D^3)$ into $M \times I$ to get D_{in}^3 , then remove νD_{in}^3 we get the same cobordism Z_1 . Then when the attaching sphere S^1 of a two handle goes through $S^1 \times S^3 \setminus D^4 \subset M_1$, by slightly perturbing the attaching region, S^1 can be made disjoint from $S^1 \times * \in S^1 \times S^3$, then in the dotted version, the S^1 can be seen in $M \setminus \nu(\partial D^3)$. And whenever S^1 goes through the belt sphere $* \times S^3$, in the dotted version, $S^1 \subset M \setminus \nu(\partial D^3)$ intersects with that D^3 . In short, the dotted version of a one handle can be described as:

Definition 3.1. For a 5-dimensional cobordism Z from $\partial_- Z$ to $\partial_+ Z = M$ with M connected, the *dotted version of describing a new cobordism* $Z_1 = Z \cup h_1$ is by choosing an embedding $\beta^\bullet : D^3 \hookrightarrow M$ with $\beta := \partial D^3$ denoting a dotted 2-sphere, then Z_1 is diffeomorphic to $Z \setminus \nu \beta_{in}^\bullet$ with $\beta_{in}^\bullet : D^3 \hookrightarrow Z$ obtained by pushing interior of $\beta^\bullet(D^3)$ into Z a bit. Whenever the attaching sphere of any 2-handle runs over the belt sphere of h_1 , it intersects β^\bullet (or equivalently, it links with β).

Running the same story of a trivial 2-handle, i.e. the attaching region is a 0-framed unknotted $S^1 = \partial D^2$ with $D^2 \hookrightarrow M$, we get the dotted version of a trivial 2-handle:

Definition 3.2. For a 5-dimensional cobordism Z from $\partial_- Z$ to $\partial_+ Z = M$ with M connected, the *dotted version of describing a new cobordism* $Z_1 = Z \cup h_2$ with h_2 a trivial 2-handle, is by

choosing an embedding $\gamma^\bullet : D^2 \hookrightarrow M$ with $\gamma := \partial D^2$ denotes as a dotted 1-sphere, then Z_1 is diffeomorphic to $Z \setminus \nu\gamma_{in}^\bullet$ with $\gamma_{in}^\bullet : D^2 \hookrightarrow Z$ obtained by pushing interior of $\gamma^\bullet(D^2)$ into Z a bit. Whenever the attaching sphere of any 3-handle runs over the belt sphere of h_2 , it intersects with γ^\bullet (or equivalently, it links with γ).

Now if we are giving *both* data from the above 2 definitions, we can perform the *dotted and 0-framed replacement*:

Proposition 3.3. *For 5-dimensional cobordism Z with $\partial_+ Z = M$ Suppose we are given $\beta^\bullet : D^3 \hookrightarrow M$, $\gamma^\bullet : D^2 \hookrightarrow M$ with β, γ the corresponding boundary embeddings. If $\gamma S^1 \cap \beta S^2 = \emptyset$, then $\partial_+(Z \setminus \nu\beta_{in}^\bullet \cup_{(\gamma,0)} h_2) = \partial_+(Z \setminus \nu\gamma_{in}^\bullet \cup_{(\beta,0)} h_3)$. Here $(\gamma, 0)$ means that the attaching sphere of h_2 is γ and the framing is induced by $\nu(\gamma, \gamma^\bullet) \oplus \nu(\gamma^\bullet, M)|_\gamma$, similar for $(\beta, 0)$.*

Proof. Both sides are $(M \setminus (\nu\beta \cup \nu\gamma)) \cup_{(\beta,0)} D^3 \times S^1 \cup_{(\gamma,0)} D^2 \times S^2$. □

Remark 3.4. Later on in the dotted version, when we say the attaching region $S^{i-1} \times D^{5-i}$ of some handle h_i ($i = 2$ or 3) is *0-framed*, we will specify a $D^i \hookrightarrow M$ with the attaching sphere $S^{i-1} = \partial D^i$, and the framing is naturally induced by $\nu(S^{i-1}, D^i) \oplus \nu(D^i, M)|_{S^{i-1}}$. Usually the D^i is clear from the context. ┘

Remark 3.5. Let $Z' = Z \setminus \nu\beta_{in}^\bullet \cup_{(\gamma,0)} h_2$ and $Z'' = Z \setminus \nu\gamma_{in}^\bullet \cup_{(\beta,0)} h_3$. Note that the two cobordisms result in the same boundary manifolds but they are different cobordisms themselves. In fact, when $Z = M \times I$, we can embed Z' into a bigger $M \times I$ such that $Z' \cup_{\partial_+ Z' = \partial_- \bar{Z}''} \bar{Z}'' = M \times I$ where \bar{Z}'' denotes the orientation-reversing cobordism from $\partial_+ Z''$ to $\partial_- Z''$ (see Figure 3.1).

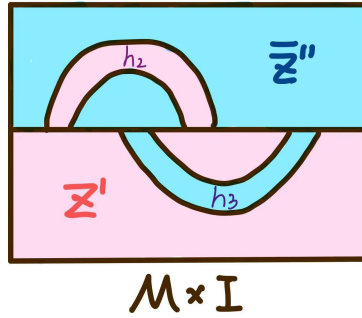


Figure 3.1 / $Z' \cup \bar{Z}'' = M \times I$

Recall what we obtained in the last chapter: For any half-unknotted barbell in M^4 , we find a loop of framed S^1 , i.e. an element in $\pi_1(\text{Emb}(S^1 \times D^3, M \# S^1 \times S^3), *)$ which represents (by the map $\text{ps}_{\nu S}$) that barbell diffeomorphism. And thus, it gives a Cerf diagram containing a single eye of (1,2)-handle pair which results in the barbell diffeomorphism. And our original goal is to find a Cerf diagram containing a single eye of (2,3)-handle pair resulting in the same barbell diffeomorphism, then calculate its Hatcher-Wagoner invariants. Therefore, the only thing we need to do is to change the eye of (1,2)-handle pair into the eye of (2,3)-handle pair. Then the

natural idea is to do the *one-parameter version of dotted and 0-framed replacement*. To do that, we need the following lemma (this appeared in the work of Gay in [6] when $M = S^4$, and there Gay was going from the opposite direction, namely, to change a certain eye of (2,3)-handle pair into an eye of (1,2)-handle pair, but the idea of the proof is totally the same):

Lemma 3.6. *Consider a one-parameter family of embeddings: $\beta_t^\bullet : D^3 \hookrightarrow M$ (denote $\beta_t = \partial\beta_t^\bullet$) and $\gamma_t : S^1 \hookrightarrow M, t \in [0, 1]$ with the following properties:*

- (1) γ_0 extends to $\gamma_0^\bullet : D^2 \hookrightarrow M$, i.e. $\gamma_0^\bullet|_{\partial D^2} = \gamma_0$, moreover, γ_0^\bullet intersects β_0 transversely at a single point.
- (2) $\beta_1^\bullet = \beta_0^\bullet$ and $\gamma_1 = \gamma_0$.
- (3) For all $t \in [0, 1]$, $\gamma_t \cap \beta_t = \emptyset$.
- (4) γ_i intersects β_i^\bullet transversely at a single point, for $i \in \{0, 1\}$.

Then there exists an extension of β_t^\bullet and γ_t to $t \in [0, 3]$ and a one-parameter family of embeddings $\gamma_t^\bullet : D^2 \hookrightarrow M, t \in [0, 3]$ satisfying:

- (1) For all $t \in [0, 1]$, γ_t and β_t^\bullet stay the same as given.
- (2) For $t \in [0, 3]$, $\partial\gamma_t^\bullet = \gamma_t$.
- (3) $\beta_3^\bullet = \beta_0^\bullet$ and $\gamma_3^\bullet = \gamma_0^\bullet$.
- (4) For all $t \in [1, 3]$, γ_t intersects β_t^\bullet transversely at a single point.
- (5) $\beta_0 = \beta_3$ intersects $\gamma_0^\bullet = \gamma_3^\bullet$ transversely at a single point.
- (6) The path $\gamma_t^\bullet, t \in [0, 3]$ is null homotopic rel $t \in \{0, 3\}$ in $\text{Emb}(D^2, M)$.

Proof. First we construct γ_t^\bullet for $t \in [0, 1]$: Since $\beta_t \cap \gamma_t = \emptyset$, choose an isotopy $\psi_t : M \rightarrow M$ which sends γ_0 to γ_t and sends β_0 to β_t , then let $\gamma_t^\bullet := \psi_t(\gamma_0^\bullet)$.

But now γ_1^\bullet may not be equal to γ_0^\bullet . Then we construct γ_t^\bullet and β_t^\bullet for $t \in [1, 2]$ such that $\gamma_0^\bullet = \gamma_2^\bullet$: Let $\phi_t : M \rightarrow M$ be the isotopy such that $\phi_t(\gamma_1^\bullet) = \gamma_{2-t}^\bullet$ then define $\gamma_t^\bullet = \phi_t(\gamma_1^\bullet)$ and $\beta_t^\bullet = \phi_t(\beta_1^\bullet)$.

But now β_2^\bullet may not be equal to β_0^\bullet . Then we construct γ_t^\bullet and β_t^\bullet for $t \in [2, 3]$ such that $\gamma_t^\bullet = \gamma_2^\bullet = \gamma_0^\bullet$ for $t \in [2, 3]$ and $\beta_3^\bullet = \beta_0^\bullet$: By condition (4), β_0^\bullet is a normal disk of γ_0 and β_1^\bullet is a normal disk of γ_1 , since $\gamma_2 = \phi_2(\gamma_1) = \gamma_0$ and $\beta_2^\bullet = \phi_2(\beta_1^\bullet)$, so β_2^\bullet is also a normal disk of $\gamma_2 = \gamma_0$. That means, both β_2^\bullet and β_0^\bullet are normal disks of γ_0 , then we can isotope β_2^\bullet to β_0^\bullet during $t \in [2, 3]$, one way is to shrink β_2^\bullet to β_t^\bullet inside β_2^\bullet to a standard normal ball of γ_0 and then expand back to β_0^\bullet .

Then from the construction one can see directly that conditions (1), (2), (3), (5) are satisfied. For condition (4), for $t \in [1, 2]$, $|\gamma_t \cap \beta_t^\bullet| = |\phi_t \gamma_1 \cap \phi_t \beta_1^\bullet| = |\gamma_1 \cap \beta_1^\bullet| = 1$, for $t \in [2, 3]$, β_t^\bullet is always a normal disk of $\gamma_t = \gamma_0$ so satisfied. For condition (6), by $\gamma_t^\bullet = \gamma_{2-t}^\bullet, t \in [0, 2]$ and $\gamma_t^\bullet = \gamma_0^\bullet, t \in [2, 3]$, then of course the condition is satisfied. \square

Proposition 3.7 (one-parameter version of dotted and 0-framed replacement). *Suppose that $\beta_t^\bullet : D^3 \hookrightarrow M, \gamma_t : S^1 \hookrightarrow M, t \in [0, 1]$ satisfying the four conditions in the above lemma, moreover, $\beta_t^\bullet = \beta_0^\bullet$, and γ_0^\bullet is a standard normal disk of β_0 , by using the above lemma, we extend β_t^\bullet and γ_t to $t \in [0, 3]$ and get γ_t^\bullet with $\gamma_t = \partial\gamma_t^\bullet$ for $t \in [0, 3]$. We further assume γ_1^\bullet and γ_0^\bullet yield the same framing for $\gamma_1 = \gamma_0$. Since $\beta_0^\bullet = \beta_3^\bullet$ and $\gamma_0^\bullet = \gamma_3^\bullet$,*

- (1) *For $t \in [0, 1]$, regard $\beta_t^\bullet = \beta_0^\bullet$ as dotted version of a trivial 1-handle, regard $\gamma_t = \partial\gamma_t^\bullet$ as a loop of 0-framed 2-handle. Then it gives $\mathcal{H}_1 \in \pi_0\mathcal{P}$ with Cerf diagram being a single eye of (1,2)-handle pair and results in $f_{\mathcal{H}_1} \in \pi_0 \text{Diff}(M, \partial)$.*
- (2) *For $t \in [0, 3]$, regard γ_t^\bullet as dotted version of a trivial 2-handle (the 2-handle can be made not moving for $t \in [0, 3]$ by condition (6) in above lemma), regard $\beta_t = \partial\beta_t^\bullet$ as a loop of framed 3-handle. Then it gives $\mathcal{H}_2 \in \pi_0\mathcal{P}$ with Cerf diagram being a single eye of (2,3)-handle pair and results in $f_{\mathcal{H}_2} \in \pi_0 \text{Diff}(M, \partial)$.*

Then $[f_{\mathcal{H}_1}] = [f_{\mathcal{H}_2}] \in \pi_0 \text{Diff}(M, \partial)$.

Proof. First consider $\beta_t^\bullet, \gamma_t = \partial\gamma_t^\bullet, t \in [0, 3]$, this loop of handle decompositions of $M \times I$ yields $\mathcal{H}_0 \in \pi_0\mathcal{P}$ with Cerf diagram being a single eye of (1,2)-handle pair. And by doing dotted and 0-framed replacement *pointwise* for $t \in [0, 3]$ from \mathcal{H}_0 , we get another pseudo-isotopy which is exactly $\mathcal{H}_2 \in \pi_0\mathcal{P}$, thus $f_{\mathcal{H}_0} = f_{\mathcal{H}_2}$.

But note that $\beta_1^\bullet = \beta_3^\bullet$ and $\gamma_1 = \gamma_3$ with the same framing induced by γ_1^\bullet and γ_3^\bullet , so for $t \in [1, 3]$, β_t^\bullet and $\gamma_t = \partial\gamma_t^\bullet$ also determines a loop of handle constructions for $M \times I$, thus a pseudo-isotopy $\mathcal{H} \in \pi_0\mathcal{P}$. But here β_t^\bullet intersects γ_t transversely at a single point for all $t \in [1, 3]$, which means they are all at cancelling position for all t , thus $[\mathcal{H}] = [\text{id}] \in \pi_0\mathcal{P}$. So $\beta_t^\bullet, \gamma_t = \partial\gamma_t^\bullet, t \in [0, 3]$ together gives a pseudo-isotopy $[\mathcal{H}_0 = \mathcal{H}_1 * \mathcal{H}] = [\mathcal{H}_1] \in \pi_0\mathcal{P}$.

Thus $[f_{\mathcal{H}_1}] = [f_{\mathcal{H}_0}] = [f_{\mathcal{H}_2}] \in \pi_0 \text{Diff}(M, \partial)$. \square

For any half-unknotted implanted barbell $\beta = (R, S, \gamma)$, from the last chapter, we get a loop of (1,2)-handle pair $g_\beta \in \pi_0\mathcal{P}$ resulting in the barbell diffeomorphism. By first changing to the dotted version and then applying Proposition 3.7, we can see that there is a natural extension $\gamma_t^\bullet, t \in [0, 1]$ by pulling γ_0^\bullet along γ_t in T_R , i.e. $\gamma_t^\bullet = \gamma_0^\bullet \cup_{s \in [0, t]} \gamma_s$. Thus $\gamma_1^\bullet = \gamma_0^\bullet \text{tube}_\gamma R$. Thus framing of $\gamma_1 = \gamma_0$ induced by γ_1^\bullet is the same as that induced by γ_0^\bullet (For moving pictures of γ_t^\bullet in the case of $\beta = \delta_k$, see Figure 3.4). Thus we have:

Corollary 3.8. *For any half-unknotted implanted barbell $\beta = (R, S, \gamma)$ in M^4 , there exists $f_\beta \in \pi_0\mathcal{P}$ whose Cerf diagram contains only a single eye of (2,3)-handle pair resulting in that barbell diffeomorphism. In particular, the first Hatcher-Wagoner invariant $\Sigma(f_\beta) = 0 \in \text{Wh}_2(\pi_1 M)$.*

Remark 3.9. We are not going into the definition of the first Hatcher-Wagoner invariant $\Sigma : \pi_0\mathcal{P} \rightarrow \text{Wh}_2(\pi_1 M)$ (to see a clear definition, read [3]), but briefly speaking, it records the handle slides that happen during $\{f_t : M \times I \rightarrow I\} \in \pi_1(\mathcal{F}, \mathcal{E})$. So when there's only a single eye of (2,3)-handle pair, there's at most one handle for each index, so no handle slides at all time. \square

Remark 3.10. Note that in the proof of the above proposition, we obtain

$$\mathcal{H}_2 = \{H_{2,t}, t \in [0, 3] \text{ which is a loop of (2,3)-handle constructions of } M \times I\}$$

by doing dotted and 0-framed replacement *pointwise* from

$$\mathcal{H}_0 = \{H_{0,t}, t \in S^1 \text{ which is a loop of (1,2)-handle construction of } M \times I\}.$$

Denote $Z_{\mathcal{H}_0}$ (resp. $Z_{\mathcal{H}_2}$) as the corresponding cobordism from $S^1 \times M$ to $Y_{\mathcal{H}} = M \times I / ((x, 0) \sim (f_{\mathcal{H}_0}(x), 1))$ (resp. $Y_{\mathcal{H}_2} = M \times I / ((x, 0) \sim (f_{\mathcal{H}_2}(x), 1))$). Since $f_{\mathcal{H}_0} = f_{\mathcal{H}_2}$, by Remark 3.5 we know that $Z_{\mathcal{H}_0} \cup \bar{Z}_{\mathcal{H}_2} = S^1 \times X \times I$, thus $0 = \Sigma(\mathcal{H}_0) + \Sigma(\bar{\mathcal{H}}_2) = \Sigma(\mathcal{H}_1) + \Sigma(\bar{\mathcal{H}}_2)$. Similarly, $\Theta(\mathcal{H}_1) + \Theta(\bar{\mathcal{H}}_2) = 0$. By [3, section 9.1] we know that there exists an involution $\bar{\cdot}$ on both $\text{Wh}_2(\pi_1 M)$ and $\text{Wh}_1(\pi_1 M; \mathbb{Z}_2 \times \pi_2 M)$ such that $\Sigma(\bar{f}) = \bar{\Sigma}(f)$, $\Theta(\bar{f}) = \bar{\Theta}(f)$, $\forall f \in \pi_0\mathcal{P}$. For implanted barbell β , we thus have $\Sigma(g_\beta) = -\bar{\Sigma}(f_\beta) = 0$, $\Theta(g_\beta) = -\bar{\Theta}(f_\beta)$. \square

Example 3.11. We apply the above procedure to Budney and Gabai's barbell diffeomorphisms δ_k , changing its corresponding loop of (1,2)-handle pair which we obtained in the last chapter into the loop of (2,3)-handle pair.

Recall the loop of (1,2)-handle constructions for Cerf diagram resulting in δ_k (again we let $k = 4$ for simplicity), as shown in Figure 3.2.

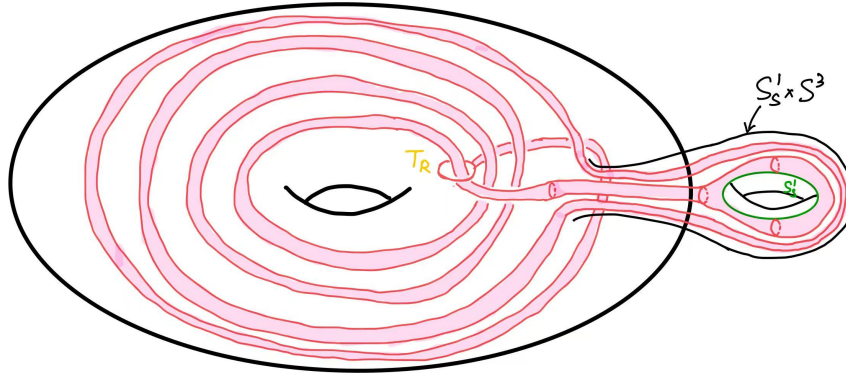


Figure 3.2 / Element in $\pi_1(\text{Emb}(S^1 \times D^3, S^1 \times D^3 \# S^1 \times S^3), *)$ which represents barbell δ_k .

Change it to the one-parameter dotted version, i.e. $\beta_t^\bullet = \beta_0^\bullet : D^3 \hookrightarrow S^1 \times D^3$, $\gamma_t : S^1 \hookrightarrow S^1 \times D^3$, $\gamma_0 = \gamma_1$ a standard meridian of $\beta_0 = \partial\beta_0^\bullet$, $t \in [0, 1]$ (see Figure 3.3).

Now following the constructions we made in Lemma 3.6, we need to find $\gamma_t^\bullet, t \in [0, 1]$. Let γ_0^\bullet just be the standard normal disk of $\partial\beta_0^\bullet$, then γ_t^\bullet is just dragging γ_0^\bullet onto T_R along the trajectory of S^1 in T_R , finally $\gamma_1^\bullet = \gamma_0^\bullet \#_{\text{tube}} R$ (see Figure 3.4).

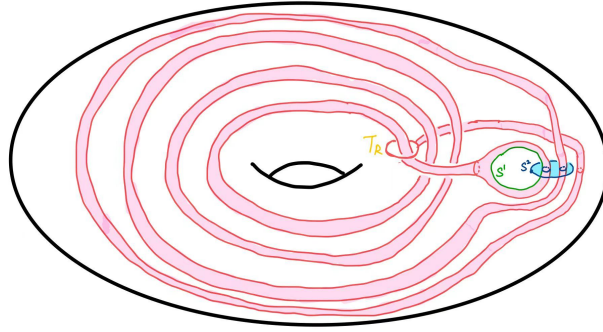


Figure 3.3 / Dotted version of the loop of (1,2)-handle pair. The dark blue is the dotted $S^2 = \partial\beta_0^\bullet$, the light blue region is $\beta_t^\bullet = \beta_0^\bullet : D^3 \hookrightarrow S^1 \times D^3$ which represents the 1-handle.

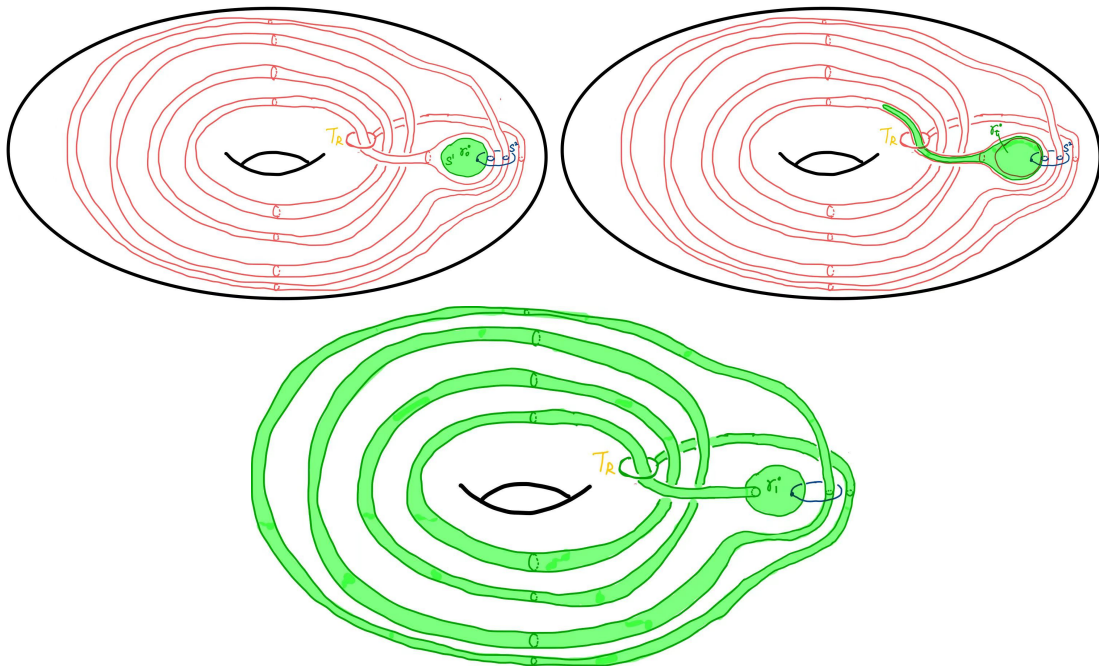


Figure 3.4 / Movement of $\gamma_t^\bullet : D^2 \hookrightarrow S^1 \times D^3$. The figures draw $\gamma_0^\bullet, \gamma_t^\bullet, \gamma_1^\bullet$.

Now at $t \in [1, 2]$, as the constructions in Lemma 3.6 read, consider $\phi_t : S^1 \times D^3 \rightarrow S^1 \times D^3$, s.t. $\gamma_t^\bullet = \phi_t(\gamma_1^\bullet) = \gamma_{2-t}^\bullet$, $\beta_t^\bullet = \phi_t(\beta_1^\bullet = \beta_0^\bullet)$. In the interval $t \in [1, 2]$, $\beta_t = \beta_0$ intersects γ_t^\bullet at a single point. While β_2^\bullet intersects γ_2^\bullet at $S^1 \cup I^1$, so when $t \in [2, 3]$, β_t shrinks in β_2^\bullet to a standard normal sphere of $\gamma_2 = \gamma_0$ so that $\beta_3 = \beta_0$. Therefore, when during $t \in [2, 3]$, β_t will intersect $\gamma_t^\bullet = \gamma_0^\bullet$ at 1 point, then at 3 points, then 1 point, which will become essential to our calculations of Hatcher-Wagoner invariants. So in order to understand how $\beta_t, t \in [2, 3]$ shrinks, we only need to understand β_2^\bullet :

β_2^\bullet is the isotopy image of $\beta_1^\bullet = \beta_0^\bullet$ when pulling γ_1 back to standard γ_0 along R , but this is just the *backward barbell diffeomorphism* with data $\beta = (R_0, S, \gamma)$. What's more, the β_1^\bullet is just a mid-ball of this implanted barbell (see Figure 3.5)! Thus in [4] we know the embedded surgery description of the mid-ball after performing a barbell. In Figure 3.6 we draw the embedded surgery explicitly to get β_2^\bullet .

Then performing *one-parameter version of dotted and 0-framed replacement* we regard $\gamma_t^\bullet, t \in [0, 3]$ as the dotted version of $h_{2,t}, t \in [0, 3]$ and regard $\beta_t, t \in [0, 3]$ as the attaching spheres of $h_{3,t}, t \in [0, 3]$. Thus we change the Cerf diagram of a (1,2)-handle pair to the Cerf diagram of a (2,3)-handle pair, both resulting in barbell δ_k .

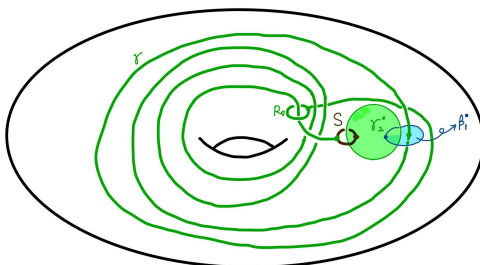


Figure 3.5 / Pulling γ_1 back to γ_0 (therefore pulling γ_1^\bullet back to γ_0^\bullet) is equivalent to doing the barbell diffeomorphism $\beta = (R_0, S, \gamma)$.

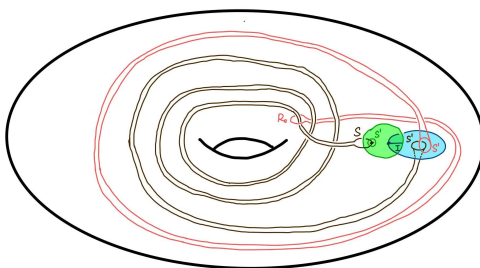


Figure 3.6 / The mid-ball β_1^\bullet becomes $\beta_2^\bullet = \partial(\beta_1^\bullet \cup h_{2,b} \cup h_{2,r})$ where $h_{2,b}$ is the 2-handle with attaching S_b^1 the brown one and core $D_b^2 = \{S \text{ tube along the brown arc}\}$, $h_{2,r}$ is the 2-handle with attaching S_r^1 the red one and core $D_r^2 = \{R \text{ tube along the red arc}\}$. As a result, $\beta_2^\bullet = (\beta_1^\bullet \setminus (S_b^1 \times D^2 \cup S_r^1 \times D^2)) \cup (D_b^2 \times S^1 \cup D_r^2 \times S^1)$. Therefore, $\beta_2^\bullet \cap \gamma_2^\bullet =$ the green S^1 and the blue I in the picture.

4 Computation for the second Hatcher-Wagoner invariant

In this chapter, first we recall the definition for the second Hatcher-Wagoner invariant, namely, $\Theta : \ker \Sigma \rightarrow \text{Wh}_1(\pi_1 M, \mathbb{Z}_2 \times \pi_2 M) / \chi(K_3 \mathbb{Z}[\pi_1 M])$. For simplicity, we restrict to the case when the Cerf diagram contains only a single eye of (2,3)-handle pair. For general definitions and well-definedness, see [1] and [3].

Recall that $\text{Wh}_1(\pi_1 M, \mathbb{Z}_2 \times \pi_2 M) = (\mathbb{Z}_2 \times \pi_2 M)[\pi_1 M] / (\beta \cdot [1], \alpha \cdot [\sigma] - \alpha^\tau \cdot [\tau \sigma \tau^{-1}], \alpha, \beta \in \mathbb{Z}_2 \times \pi_2 M, \tau, \sigma \in \pi_1 M)$ (see [3] for details).

To define Θ , we need the following data: For $\{f_t, t \in I\} \in \pi_1(\mathcal{F}, \mathcal{E})$ with Cerf diagram only a single eye of (2,3)-handle pair, suppose that the eye appears just before $t = \epsilon$ at critical value $\frac{1}{2}$, and ends right after $t = 1 - \epsilon$ at the same critical value, then consider $V := \bigcup_{t \in [\epsilon, 1-\epsilon]} f_t^{-1}(1/2) = M' \times [\epsilon, 1 - \epsilon]$ where $M' = M \# S^2 \times S^2$. For each $f_t, t \in [\epsilon, 1 - \epsilon]$, we call the belt 2-sphere of the 2-handle B_t , the attaching 2-sphere of the 3-handle A_t . Let $A := \bigcup_{t \in [\epsilon, 1-\epsilon]} A_t \cong S^2 \times I \subset V$, $B := \bigcup_{t \in [\epsilon, 1-\epsilon]} B_t \cong S^2 \times I \subset V$. Without loss of generality, we may assume that all intersections among A, B, V are transverse. Then we know that $A \cap B = I \sqcup_{i=1, \dots, k} S^1$, we denote the i -th S^1 as S_i^1 . Now, choose a base point $* \in X \times 0 \subset X \times I$, a fixed $*' \in I \subset A \cap B$ and a path $\delta_0 \subset X \times I : * \rightarrow *'$. For each i , choose a point $p_i \in S_i^1$, and two paths: $\delta_i^A \subset A : *' \rightarrow p_i$, $\delta_i^B \subset B : *' \rightarrow p_i$. Denote $\gamma_i := \delta_i^B * (\delta_i^A)^{-1} \in \pi_1(X \times I, *')$ and $\gamma_i^* := \gamma_i^{\delta_0} \in \pi_1(X \times I, *)$ by pulling γ_i from the base point $*'$ back to the base point $*$ along δ_0 . Also note that both A and B are simply connected, then choose $D_i^A \looparrowright A$ such that $\partial D_i^A = S_i^1$, $D_i^B \looparrowright B$ such that $\partial D_i^B = S_i^1$, so denote $\beta_i := [D_i^A \cup D_i^B] \in \pi_2(X \times I, p_i)$ and $\beta_i^* := \beta_i^{\delta_0 * \delta_i^B} \in \pi_2(X \times I, *)$ by pulling β_i from base point p_1 back to base point $*$ along $\delta_0 * \delta_i^B$.

To define the \mathbb{Z}_2 component, we consider two framings of the same \mathbb{R}^2 -vector bundle on S_i^1 . Consider $\nu(S_i^1, B)$ (the normal bundle of S_i^1 in B), since A intersects B transversely in V , then $E = \nu(S_i^1, B) = \nu(A, V)|_{S_i^1}$ is a 2-dimensional trivial vector bundle over S_i^1 . We define two framings on E : $e_{i,B}$ is the framing naturally induced by $D_i^B \looparrowright B$, $e_{i,A}$ is induced by the attaching data for the 3-handle, namely, $A_t \times D^2 \hookrightarrow M' = M'_t$. Then $e_{i,A} - e_{i,B} \in \pi_1(SO_2) = \mathbb{Z}$, we define $s_i := e_{i,A} - e_{i,B} \pmod{2}$. Thus for every i , we have defined $s_i \in \mathbb{Z}_2$.

Using all those notations above, we can define the second Hatcher-Wagoner invariant Θ for $F := \{f_t, t \in I\} \in \pi_1(\mathcal{F}, \mathcal{E})$:

Definition 4.1. For $F := \{f_t, t \in I\} \in \pi_1(\mathcal{F}, \mathcal{E})$ with Cerf diagram only a single eye of (2,3)-handle pair, define $\Theta(F) := \sum_{i=1, \dots, k} (s_i, \beta_i^*) \cdot [\gamma_i^*]$.

Example 4.2. Now we calculate the second Hatcher-Wagoner invariant for implanted barbell diffeomorphism δ_k :

Using results in last chapter, by projecting the movements to a single $S^1 \times D^3 \# S^1 \times S^3$, and then using the dotted version and *one-parameter version of dotted and 0-framed replacement* in $S^1 \times D^3$, we use β_t and γ_t^\bullet in $S^1 \times D^3$ to represent A_t, B_t in $S^1 \times D^3 \# S^1 \times S^3$. We see that the attaching sphere $A_t = \beta_t$ intersects $B_t = \gamma_t^\bullet \cup D^2$ at a single point for $t \in [0, 2]$ and $\bigcup_{t \in [2, 3]} A_t \cap B_t = \beta_2^\bullet \cap \gamma_2^\bullet = I \sqcup S^1$ with I the blue arc in Figure 3.6, and S^1 the green one in Figure 3.6. Thus from Figure 3.6 we see $\gamma^* = \gamma_1^* = (k-1)^* = k-1 \in \pi_1(S^1 \times D^3, *) = \mathbb{Z}$.

To calculate β^* , we need to find $D^B \looparrowright \gamma_2^\bullet \times I$ and $D^A \looparrowright \beta_2^\bullet$ which bound $\partial D^A = \partial D^B = S^1$. By projecting to a single $S^1 \times D^3$ for $t \in [2, 3]$, we know that $\gamma_t^\bullet = \gamma_2^\bullet$ is the standard green one in Figure 3.6 which won't change, so it's enough to find $D^B \looparrowright \gamma_2^\bullet$. For D^B , we have an immediate choice in Figure 3.6, which is just the small disk in green γ_2^\bullet which bounds the green S^1 . For D^A , we pull the green S^1 , which is the meridian of the brown tube, along the black tube back to the meridian of brown $S^1 \subset$ the blue D^3 , which is just a parallel of red $S^1 \subset$ the blue D^3 , then capping with the red D^2 which is the red long tube connected with R_0 . In all, $\beta^* = \beta_1^* = (R_0^\gamma)^* \in \pi_2(S^1 \times D^3, *) = 0$, here γ is the arc in the definition of barbell, which is the arc connecting S with R_0 .

Now we calculate $s = s_1$. For $s_1^A =$ attaching framing of $\nu(A, V)|_{t \in [2, 3], S^1 \subset A}$, from the dotted version we know that $\{A_t | t \in [2, 3]\} \subset \beta_2^\bullet$ is 0-framed, thus s_1^A has a chapter which is the normal bundle of β_2^\bullet in $S^1 \times D^3$, which is exactly a normal chapter of $\nu(S^1, \gamma_2^\bullet) \subset \nu(S^1, \gamma_2^\bullet \times I = B|_{t \in [2, 3]})$ when restricted to the green S^1 . Thus $s^A = s^B$, $s = 0$.

Thus we obtain that $\Theta(\delta_k) = (0, (R_0^\gamma)^*[k-1]) = (0, 0)[k-1] = 0 \in \text{Wh}_1(\mathbb{Z}, \mathbb{Z}_2 \times 0)$.

The calculation can be generalized similarly to any half-unknotted implanted barbell $\beta = (R_0, S, \gamma)$ with S unknotted. After doing an isotopy of M we can put $S = \partial\beta_0^\bullet$ into the standard position and use strategies in Chapter 3 to find a Cerf diagram containing a single eye of $(2, 3)$ -handle pair with $\gamma_t^\bullet, \beta_t, t \in [0, 3]$. Note that for $t \in [0, 2]$, β_t intersects γ_t^\bullet at a single point, so as the example suggests, the second Hatcher-Wagoner invariant can be calculated just using β_2^\bullet and $\gamma_2^\bullet = \gamma_0^\bullet$. Assume we have obtained $\beta_2^\bullet \subset M$, $\beta_2^\bullet \cap \gamma_2^\bullet = I \sqcup_{i=1, \dots, k} S^1$, then the same argument of the example works in general case, i.e. using notations above, we have

$$\Theta(F) = \sum_{i=1, \dots, k} (s_i, \beta_i^*) \cdot [\gamma_i^*]$$

where for each i , $\gamma_i = \delta_i^B * (\delta_i^A)^{-1}$ with $\delta_i^A \looparrowright \beta_2^\bullet$, $\delta_i^B \looparrowright \gamma_2^\bullet$ connecting $*'$ and $p_i \in S_i^1$, $\beta_i = D_i^A \cup D_i^B$ with $D_i^A \looparrowright \beta_2^\bullet$ and $D_i^B \looparrowright \gamma_2^\bullet$, $s_i = e_{i,A} - e_{i,B} \bmod 2$ with $E = \nu(S_i^1, \gamma_2^\bullet \times I) = \nu(\beta_2^\bullet, M \times I)|_{S_i^1}$. $\nu(S_i^1, \gamma_2^\bullet \times I) = \nu(S_i^1, \gamma_2^\bullet) \oplus E' = \nu(S_i^1, D_i^B) \oplus E'$, $\nu(\beta_2^\bullet, M \times I)|_{S_i^1} = \nu(\beta_2^\bullet, M) \oplus E''$ where E', E'' are two 1-dimensional trivial bundle over S_i^1 . $e_{i,B}$ is the framing induced by $\nu(S_i^1, \gamma_2^\bullet)$ and $e_{i,A}$ is the framing induced by $\nu(\beta_2^\bullet, M)|_{S_i^1}$. But since β_2^\bullet intersects γ_2^\bullet transversely at S_i^1 in M , so $\nu(S_i^1, \gamma_2^\bullet) = \nu(\beta_2^\bullet, M)|_{S_i^1}$, thus, $s_i = e_{i,A} - e_{i,B} = 0$.

To calculate β_i^* and γ_i^* , we need a smarter change: It can be very hard to visualize β_2^\bullet , especially when R_0 goes through β_0^\bullet in a strange way ($R_0 \cap \beta_0^\bullet$ may be knotted in $D^3 = \beta_0^\bullet$).

But we know that there is an isotopy $\phi_2 : M \rightarrow M$ which is isotopic to identity such that $\phi_2(\beta_1^\bullet = \beta_0^\bullet) = \beta_2^\bullet$, $\phi_2(\gamma_1^\bullet) = \gamma_2^\bullet$. This means we can pullback every data we need to β_1^\bullet and γ_1^\bullet . Then it will be easier to visualize in the given implanted barbell β .

Note that $\beta_1^\bullet = \beta_0^\bullet$ is the given unknotted ball which bounds S , i.e. $S = \partial\beta_1^\bullet$. Also recall that $\gamma_1^\bullet = \gamma_0^\bullet \text{ tube}_\gamma R_0$. Without loss of generality, we may assume that $\beta_0^\bullet \cap R_0 = \sqcup_{i=1, \dots, k} S_i^1$, $\text{int}(\beta_0^\bullet) \cap \gamma = \emptyset$ (by finger-pushing R_0 along γ without reaching S , such that the arc γ of the implanted barbell β can be made arbitrarily short). Let $*_0 = S \cap \gamma$ be the base point, note that $*_0$ will not move during the whole time $t \in [0, 3]$ (in Figure 3.6, $*_0$ is the left vertex of the blue I). For each i , find a path $\delta_i^B \subset \beta = (R_0, S, \gamma)$ which is a path from $*_0$ to $p_i \in S_i^1$. Also, for each i , S_i^1 divides R_0 into two embedded disk D_i and D'_i , where D'_i is the one connected to the arc γ . Let $D_i^B = D_i$. Now comes the following main theorem:

Theorem 4.3. *For a half-unknotted implanted barbell $\beta = (R_0, S, \gamma)$ with $S = \partial\beta_0^\bullet$ where $\beta_0^\bullet : D^3 \hookrightarrow M$, by finger-pushing R_0 along the arc, we can make γ short enough such that $\text{int}(\beta_0^\bullet) \cap \gamma = \emptyset$. Now suppose that $\beta_0^\bullet \cap R_0 = \sqcup_{i=1}^k S_i^1$. Choose $p_i \in S_i^1$ and let $*_0 = \gamma \cap S$ be the base point. For each i , find a path $\delta_i^B \subset \beta = (R_0, S, \gamma)$ which is a path from $*_0$ to $p_i \in S_i^1$. Also, for each i , S_i^1 divides R_0 into two embedded disks D_i and D'_i , where D'_i is the one connected to the arc γ . Let $D_i^B = D_i$. Then the $f_\beta \in \pi_0\mathcal{P}$ we constructed in Corollary 3.8 which results in the implanted barbell diffeomorphism w.r.t. β satisfies:*

$$\Theta(f_\beta) = \sum_{i=1, \dots, k} (0, [D_i^B]^{\delta_i^B}) \cdot [\delta_i^B]$$

Here we identify $\pi_i(M, *_0)$ with $\pi_i(M, \beta_0^\bullet)$ so that $[D_i] \in \pi_2(M, \beta_0^\bullet)$ and $\delta_i^B \in \pi_1(M, \beta_0^\bullet)$.

Proof. In the above discussion we already showed that $s_i = 0 \in \mathbb{Z}_2$ for all i , we only need to show that $[D_i]^{\delta_i^B} = \beta_i^*$ and $[\delta_i^B] = \gamma_i^*$. But since we choose $*_0 \in M$ which won't move during the whole procedure, then $* = (*_0, 0) \in M \times 0$, $*' = (*_0, 1/2) \in (M')_t = M \# S^2 \times S^2$. So for the calculation, we can choose β_i^* and γ_i^* based at $*_0 \in M$. Then by definition and by changing data to $\gamma_1^\bullet = \gamma_0^\bullet \text{ tube}_\gamma R_0$ and $\beta_1^\bullet = \beta_0^\bullet$, we pick $\delta_i^A \subset \beta_1^\bullet$ connecting $*_0$ with p_i , pick $D_i^A \looparrowright \beta_1^\bullet$ with $\partial D_i^A = S_i^1$. Then by definition $\beta_i^* = [\phi_2((D_i^A \cup D_i^B)^{\delta_i^B})] = [(D_i^A \cup D_i^B)^{\delta_i^B}] \in \pi_2(M, *_0)$, since by Lemma 3.6, $\phi_t : M \rightarrow M, t \in [1, 2]$ is a one-parameter family of isotopies of M with $\phi_1 = \text{id}$ and $\phi_t(*_0) = *_0, \forall t \in [1, 2]$. But β_0^\bullet is an embedded contractible D^3 , so $\beta_i^* = [D_i^B]^{\delta_i^B}$. Also for the same reason, $\gamma_i^* = \delta_i^B * (\delta_i^A)^{-1} \in \pi_1(M, *_0)$ which is $\delta_i^B \in \pi_1(M, \beta_0^\bullet)$. \square

Example 4.4. Using Figures 3.3 and 3.4 (where we draw γ_1^\bullet and $\beta_1^\bullet = \beta_0^\bullet$ explicitly when $\beta = \delta_k$ in $S^1 \times D^3$) and results in Theorem 4.3, one can easily see that for the implanted barbell δ_k in $S^1 \times D^3$, $\Theta(f_{\delta_k}) = (0, 0) \cdot [k - 1] = 0$, which coincides with Example 4.2.

5 Generalizations to immersed barbell diffeomorphisms

In this chapter we generalize implanted barbell diffeomorphisms to *immersed barbell diffeomorphisms* from the perspective of graspers, then we show that half-unknotted immersed barbell diffeomorphisms are pseudo-isotopic to identity, and we can similarly find $f_\beta \in \pi_0\mathcal{P}$ resulting in that immersed barbell diffeomorphism such that its Cerf diagram contains a single eye of (2,3)-handle pair. Then we compute the Hatcher-Wagoner invariants for f_β . As a corollary, we show that for every $\sigma \in \pi_2 M$ and $\forall \gamma \in \pi_1 M$, $(0, \sigma) \cdot [\gamma] \in \text{Wh}_1(\pi_1 M, \mathbb{Z}_2 \times \pi_2 M)$ can be realized by a half-unknotted immersed barbell diffeomorphism, i.e. there exists $f \in \pi_0\mathcal{P}$ resulting in a half-unknotted immersed barbell diffeomorphism such that $\Sigma(f) = 0, \Theta(f) = (0, \sigma) \cdot [\gamma]$.

Definition 5.1. An *immersed barbell* $\beta = (R_0, S, \gamma)$ consists of a framed immersed 2-sphere R_0 in M , a framed embedded 2-sphere S in M , and a framed embedded arc γ in M connecting R_0 and S . By finger-pushing R_0 along γ into S to get another framed immersed 2-sphere R in M such that $R \cap S = 2$ pts, we do surgery along νS to get $M_{\nu S}$, then after surgery we get $T_R = (R \setminus D^2 \times \partial I) \cup S^1 \times I$ which is a framed immersed torus. As in Chapter 2, νT_R also defines an element $[\nu T_R] \in \pi_1(\text{Emb}(\nu S^1, M_{\nu S}), *)$. We define the corresponding *immersed barbell diffeomorphism* as $\text{ps}_{\nu S}[\nu T_R]$, where we define $\text{ps}_{\nu S}$ in Definition 2.2.

In the same manner of Proposition 2.4 we can show that, when S is unknotted, there is a loop $g_\beta \in \pi_0\mathcal{P}$ of handle constructions of $M \times I$ containing a single (1,2)-handle pair which results in that immersed barbell diffeomorphism. And by doing *one-parameter version of dotted and 0-framed replacement* stated in Lemma 3.6 we change the loop of (1,2)-handle pair g_β to a loop of (2,3)-handle pair $f_\beta \in \pi_0\mathcal{P}$, thus we can compute the Hatcher-Wagoner invariants for f_β .

Throughout the process, the only difference from the embedded case is that, to obtain $\gamma_t^\bullet, t \in [0, 1]$ with $\gamma_t = \partial \gamma_t^\bullet$ which is the given loop of attaching spheres of the 2-handles, in the previous embedded case, we naturally choose $\gamma_t^\bullet = \gamma_0^\bullet \cup_{s \in [0, t]} \gamma_s$ (that is, pull γ_0^\bullet along $\gamma_t, t \in [0, 1]$), but now since T_R is an immersed torus, it may have double self-intersection points, then there'll be $t_0 < t_1$ such that $p := \gamma_{t_0} \cap \gamma_{t_1} \neq \emptyset$ (by a reparametrization of T_R , which won't change the resulting $g_\beta \in \pi_0\mathcal{P}$, one can assume that for any $t_0 \neq t_1$, there's at most one intersection point between γ_{t_0} and γ_{t_1}). In this case when the intersection point is reached for the second time at $t = t_1$, one must finger-push the small intersection disk $D^2 = \{\text{normal disk of } T_R \text{ at point } p\} \subset \gamma_t^\bullet, t \in (t_0, t_1)$ towards the trajectory of γ_t to get an embedded $\gamma_t^\bullet, t > t_1$. Figure 5.1 is an illustration.

After understanding that we can deduce the following theorem, which is a generalization of Theorem 4.3:

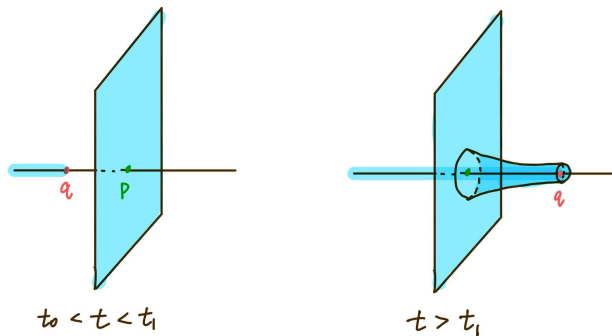


Figure 5.1 / Here we draw a local region of T_R near the self-intersection point in an \mathbb{R}^3 -slice. $p = \gamma_{t_0} \cap \gamma_{t_1}$, $q \in \gamma_t$, and the blue region is the local part of γ_t^\bullet in a \mathbb{R}^3 -slice. When $t > t_1$, the blue disk $\subset \gamma_t^\bullet$ will be finger-pushed along γ_t to ensure γ_t^\bullet is embedded.

Theorem 5.2. For a half-unknotted immersed barbell $\beta = (R_0, S, \gamma)$ with $\beta_0^\bullet : D^3 \hookrightarrow M$, $S = \partial\beta_0^\bullet$, Perturb self-intersections of R_0 away from β_0^\bullet . By finger-pushing R_0 along the arc, we can make γ short enough such that $\text{int}(\beta_0^\bullet) \cap \gamma = \emptyset$. Now suppose that $\beta_0^\bullet \cap R_0 = \sqcup_{i=1, \dots, k} S_i^1$. Choose $p_i \in S_i^1$ and let $*_0 = \gamma \cap S$ be the base point. For each i , find a path $\delta_i^B \subset \beta = (R_0, S, \gamma)$ which is a path from $*_0$ to $p_i \in S_i^1$. Also, for each i , S_i^1 divides R_0 into two embedded disks D_i and D'_i , where D'_i is the one connected to the arc γ . Let $D_i^B = D_i$. Then the $f_\beta \in \pi_0\mathcal{P}$ we constructed which results in the immersed barbell diffeomorphism w.r.t. β satisfies:

$$\Theta(f_\beta) = \sum_{i=1, \dots, k} (0, [D_i^B]^{\delta_i^B}) \cdot [\delta_i^B]$$

Here we identify $\pi_i(M, *_0)$ with $\pi_i(M, \beta_0^\bullet)$ so that $[D_i] \in \pi_2(M, \beta_0^\bullet)$ and $\delta_i^B \in \pi_1(M, \beta_0^\bullet)$.

Proof. By finger-pushing R_0 along γ into S to get R , we then attach a 1-handle, in the dotted version, we push β_0^\bullet into $M \times I$ and cut out a neighborhood of it. On the top we just do surgery on νS and get T_R which is an immersed framed torus in $M \# S^1 \times S^3$. On the left side of Figure 5.2 we draw the immersed T_R in the dotted version with standard γ_0^\bullet in green. Thus we get initial data $\gamma_0^\bullet, \{\gamma_t, t \in [0, 1]\} \in \pi_1(\text{Emb}(\nu S^1, M \# S^1 \times S^3), *)$ and $\beta_t^\bullet = \beta_0^\bullet, t \in [0, 1]$. To use *one-parameter version of dotted and 0-framed replacement* and calculate $\Theta(f_\beta)$ we only need to find $\gamma_t^\bullet, t \in [0, 1]$.

As in the process from the last chapter, we have $s_i = 0$, β_i^* and γ_i^* can be calculated in γ_1^\bullet and $\beta_1^\bullet = \beta_0^\bullet$. By the discussions just above the theorem, we isotope γ_0^\bullet along γ_t to γ_t^\bullet . Whenever we reach a self-intersection point $p = \gamma_{t_1}(v)$ ($\gamma_t : S^1 \rightarrow T_R \subset M : w \rightarrow \gamma_t(w)$) the second time, the normal disk at the intersection point needs to be stretched along $l_p := \cup_{t \in [t_1, 1]} \gamma_t(v)$. Thus in the case T_R has just a single self-intersection point, $\gamma_1^\bullet = (\gamma_0^\bullet \text{ tube}_\gamma R_0 - D^2) \cup S^1 \times l_p \cup D_1^2$ where $S^1 \times l_p = SN(T_R)|_{l_p}$ (for X immersed in M , we use $SN(X)$ to denote the sphere bundle of $N(X)$ which is the normal bundle of X in M), that is, each S^1 is a meridian sphere, and $D_1^2 \subset S^2 = SN(\gamma_1)|_{\gamma_1(v)}$ which bounds $S^1 = SN(T_R)|_{\gamma_1(v)}$ such that $D_1^2 \cap \gamma_0^\bullet = \emptyset$ (see Figure 5.2 for an illustration).

When there are more double self-intersection points, by a nice reparametrization of T_R , we may assume the self-intersection points are $p_j = \gamma_{t_{0,j}}(w_j) \cap \gamma_{t_{1,j}}(v_j)$, $t_{0,j} < t_{1,j}$, $j = 1, \dots, n$ and $\{v_j\}$ are distinct from each other. Then let $l_{p_j} := \cup_{t \in [t_{1,j}, 1]} \gamma_t(v_j)$. Then $\gamma_1^\bullet = (\gamma_0^\bullet \text{ tube}_\gamma R_0 -$

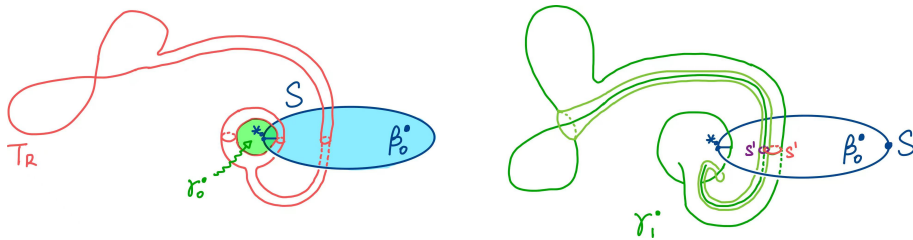


Figure 5.2 / Here we illustrate immersed torus T_R and the standard γ_0^\bullet on the left and the desired γ_1^\bullet on the right. If T_R is embedded, $\gamma_1^\bullet \cap \beta_0^\bullet = I \cap S^1$ where I is in blue and S^1 is in red. But since T_R has a self-intersection point, by finger-pushing a small disk near the intersection point along γ_t as we said in Figure 5.1, this induces another $S^1 \subset \gamma_1^\bullet \cap \beta_0^\bullet$ in purple, which is exactly a meridian of the red S^1 .

$\sqcup_{j=1, \dots, n} D^2) \cup_{j=1, \dots, n} (S^1 \times l_{p_j}) \cup_{j=1, \dots, n} D_{1,n}^2$ where $S^1 \times l_{p_j} = SN(T_R)|_{l_{p_j}}$ and $D_{1,j}^2 \subset S^2 = SN(\gamma_1)|_{\gamma_1(v_j)}$ which bounds $S^1 = SN(T_R)|_{\gamma_1(v_j)}$ such that $D_j^2 \cap \gamma_0^\bullet = \emptyset$.

Then consider $\gamma_1^\bullet \cap \beta_1^\bullet$, it's $R \cap \beta_1^\bullet$ which are k disjoint S^1 , namely $S_i^1, i = 1, \dots, k$, and several meridians of S_i^1 in β_0^\bullet . But for every meridian $S^1 = SN(T_R)|_{x_m}, x_m = \gamma_{t_{j,m}}(v_j) \in l_{p_j}, t_{j,m} \in [t_{1,j}, 1]$ for some j (like the one in purple in Figure 5.2), $D^B = (SN(T_R)|_{l_{p_j}[t_{j,m}, 1]} \cup D_{1,j}^2) \subset \gamma_1^\bullet$ which bounds that S^1 is obviously null in $\pi_2(M, \beta_0^\bullet)$, since it bounds $D^3 = DN(T_R)|_{l_{p_j}[t_{j,m}, 1]} \cup D_{1,j}^3, D_{1,j}^3 \subset D^3 = DN(\gamma_1)|_{\gamma_1(v_j)}$ where $DN(T_R)$ denotes the normal disk bundle of T_R . Thus in the calculation of Θ , every meridian S^1 corresponds to $(0, 0) \cdot [\alpha] = 0 \in Wh_1(\pi_1 M; \mathbb{Z}_2 \times \pi_2 M)$, so they make no contributions. So $\Theta(f_\beta) = \sum_{i=1, \dots, k} (0, \beta_i^*) \cdot [\gamma_i^*]$, but for the same reason, $\beta_i^* = [D_i^B]^{\delta_i^B}$, that is, stretching the disk near the intersection point along l_{p_j} makes no difference in $\pi_2(M) = \pi_2(M, \beta_0^\bullet)$. Thus, $\Theta(f_\beta) = \sum_{i=1, \dots, k} (0, [D_i^B]^{\delta_i^B}) \cdot [\delta_i^B]$. \square

Corollary 5.3. *For any $\sigma \in \pi_2 M$ with $w_2^M(\sigma) = 0, \forall \alpha \in \pi_1 M$, there is a half-unknotted immersed barbell $\beta = (R, S, \gamma)$ and $f_\beta \in \ker \Sigma \subset \pi_0 \mathcal{P}$ with its Cerf diagram being a single eye of $(2, 3)$ -handle pair resulting in the immersed barbell diffeomorphism w.r.t. β such that $\Theta(f_\beta) = (0, \sigma) \cdot [\alpha]$.*

Proof. For σ satisfying the condition, we first find R which is an immersed 2-sphere with $q \in R$, such that R represents $\sigma \in \pi_2(M, q)$. Since $w_2^M(\sigma) = 0$, we can induce some interior twists (see [9, section 1.3]) locally on R away from q to get R_σ which is a framed immersed 2-sphere representing the same $\sigma \in \pi_2(M, q)$. Then find a small embedded $\beta_0^\bullet : D^3 \hookrightarrow M$ away from R_σ with base point $*_0 \in S := \partial \beta_0^\bullet$. Fix $p \in \text{int}(\beta_0^\bullet)$. Find a path γ_1 from $*_0$ to p such that $\text{int}(\gamma_1) \cap (\beta_0^\bullet \sqcup R_\sigma) = \emptyset$, γ_1 represents $\alpha \in \pi_1(M, \beta_0^\bullet)$. Find γ_2 from p to q such that $\text{int}(\gamma_2) \cap (\beta_0^\bullet \sqcup R_\sigma) = \emptyset$, $(\gamma_1 * \gamma_2)^{-1}$ induces a natural isomorphism from $\pi_2(M, q)$ to $\pi_2(M, *_0)$: $\tau \rightarrow \tau^{(\gamma_1 * \gamma_2)^{-1}}$, we need $\sigma^{(\gamma_1 * \gamma_2)^{-1}} = \sigma$. Let $\gamma = \gamma_1 * \gamma_2$, then let $\beta = (R_\sigma, S, \gamma)$. By the above theorem, $\Theta(f_\beta) = (0, \sigma) \cdot [\alpha]$. \square

When there is an odd 2-sphere $\tau \in \pi_2 M$ (i.e. $w_2^M(\tau) = 1$), we can do more (the key point is that the π_2 component of Θ only depends on D_i , not on D'_i):

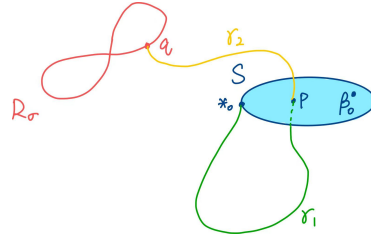


Figure 5.3 / An illustration of $\beta = (R_\sigma, S, \gamma = \gamma_1 * \gamma_2)$

Corollary 5.4. *If there exists $\tau \in \pi_2 M$ with $w_2^M(\tau) = 1$, then for any $\sigma \in \pi_2 M$, $\forall \alpha \in \pi_1 M$, there is a half-unknotted immersed barbell $\beta = (R, S, \gamma)$ and $f_\beta \in \ker \Sigma \subset \pi_0 \mathcal{P}$ with its Cerf diagram being a single eye of $(2,3)$ -handle pair resulting in the immersed barbell diffeomorphism w.r.t. β such that $\Theta(f_\beta) = (0, \sigma) \cdot [\alpha]$.*

Proof. For $\sigma \in \pi_2 M$ with $w_2^M(\sigma) = 0$, the result just follows from Corollary 5.3. For $w_2^M(\sigma) = 1$. We find an immersed 2-sphere R_σ representing σ , and an immersed 2-sphere R_τ representing τ . Connect R_τ and R_σ with a tube along an embedded arc γ_0 disjoint from both R_σ and R_τ to get $R(\gamma_0)$. Then $R(\gamma_0)$ can be made into a framed immersed 2-sphere by inducing some interior twists locally. Find a small normal disk β_0^\bullet on γ_0 , let $*_0 \in S = \partial \beta_0^\bullet$ be the base point. Find another embedded path γ from $*_0$ to $p_\tau \in R_\tau$ away from β_0^\bullet and $R(\gamma_0)$. See Figure 5.4 for an illustration. By suitably choosing γ_0 and γ , we can find $\beta = (R(\gamma_0), S, \gamma)$ such that $\Theta(f_\beta) = (0, \sigma) \cdot [\alpha]$.

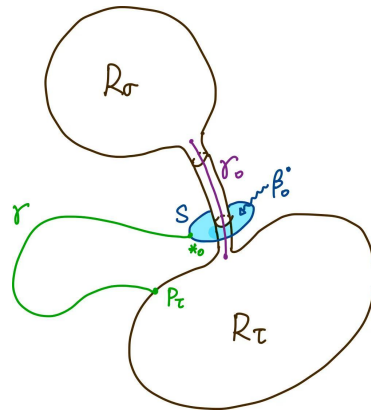


Figure 5.4 / An illustration for the proof of Corollary 5.4

□

Combining the above two corollaries we get:

Corollary 5.5. *For any $\sigma \in \pi_2 M$, $\forall \alpha \in \pi_1 M$, there is a half-unknotted immersed barbell $\beta = (R, S, \gamma)$ and $f_\beta \in \ker \Sigma \subset \pi_0 \mathcal{P}$ with its Cerf diagram being a single eye of $(2,3)$ -handle pair resulting in the immersed barbell diffeomorphism w.r.t. β such that $\Theta(f_\beta) = (0, \sigma) \cdot [\alpha]$.*

Singh proved in [3] that for any compact M^4 ,

$$W = \langle (s, \sigma) \cdot [\gamma], s = 0 \text{ or } w_2^M(\sigma) \neq 0 \rangle \subset \text{Wh}_1(\pi_1 M; \mathbb{Z}_2 \times \pi_2 M),$$

we have $W \subset \Theta(\ker \Sigma)$. In particular, when $M = X^3 \times I$, $W = \langle (0, \sigma) \cdot [\gamma] \rangle \subset \text{Wh}_1(\pi_1 M; \mathbb{Z}_2 \times \pi_2 M)$ since all $\sigma \in \pi_2 M$ have $w_2^M(\sigma) = 0$. Then,

Corollary 5.6. *For $M = X^3 \times I$, all Hatcher-Wagoner invariants Θ realizable by Singh's procedure can be realized by compositions of f_β , where β is an immersed half-unknotted barbell.*

In particular, for $M = (X_1 \# X_2) \times I$ with X_i closed, orientable, aspherical 3-manifold, Singh used W and approximations for $K_3(\mathbb{Z}\pi_1 M)$ and $\Theta(\mathcal{J} \cap \ker \Sigma)$ to show that there's $K \subset \pi_0 \text{Diff}_{PI}(M, \partial)$ and a surjection $K \rightarrow \bigoplus_{i \in \mathbb{N}} \mathbb{Z}$. Now by the above corollary, we have:

Corollary 5.7. *Let $M = (X_1 \# X_2) \times I$ with X_i closed, orientable, aspherical 3-manifold. In this case all $\sigma \in \pi_2 M = \mathbb{Z}[\pi_1 X_1 * \pi_1 X_2]$ can be realized by embedded S^2 with $w_2^M(\sigma) = 0$. Then*

$$\langle \text{implanted half-unknotted barbell diffeomorphisms} \rangle \subset \pi_0 \text{Diff}_{PI}(M, \partial)$$

is infinitely generated and of infinite \mathbb{Z} -rank.

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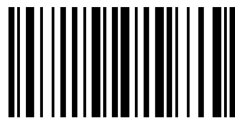
A career in mathematics research is about enduring frustrations with tenacity, being willing to be stuck on a problem, having an open mind and trying things others haven't. The path is not easy, but it is incredibly rewarding in the end.

— David Gabai

A handwritten signature in black ink, appearing to read "John Doe". The signature is highly stylized and cursive, with the first name "John" and last name "Doe" clearly visible.

A handwritten signature in black ink, appearing to read "David Gabai". The signature is highly stylized and cursive, with the first name "David" and last name "Gabai" clearly visible.

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